Chapter 6

ELECTROSTATIC BOUNDARY-VALUE PROBLEMS

Our schools had better get on with what is their overwhelmingly most important task: teaching their charges to express themselves clearly and with precision in both speech and writing; in other words, leading them toward mastery of their own language. Failing that, all their instruction in mathematics and science is a waste of time.

—JOSEPH WEIZENBAUM, M.I.T.

6.1 INTRODUCTION

The procedure for determining the electric field $\mathbf{E}$ in the preceding chapters has generally been using either Coulomb's law or Gauss's law when the charge distribution is known, or using $\mathbf{E} = -\nabla V$ when the potential V is known throughout the region. In most practical situations, however, neither the charge distribution nor the potential distribution is known.

In this chapter, we shall consider practical electrostatic problems where only electrostatic conditions (charge and potential) at some boundaries are known and it is desired to find $\mathbf{E}$ and V throughout the region. Such problems are usually tackled using Poisson's[1] or Laplace's[2] equation or the method of images, and they are usually referred to as *boundary-value* problems. The concepts of resistance and capacitance will be covered. We shall use Laplace's equation in deriving the resistance of an object and the capacitance of a capacitor. Example 6.5 should be given special attention because we will refer to it often in the remaining part of the text.

6.2 POISSON'S AND LAPLACE'S EQUATIONS

Poisson's and Laplace's equations are easily derived from Gauss's law (for a linear material medium)

$$\nabla \cdot \mathbf{D} = \nabla \cdot \varepsilon \mathbf{E} = \rho_v \tag{6.1}$$

[1]After Simeon Denis Poisson (1781–1840), a French mathematical physicist.

[2]After Pierre Simon de Laplace (1749–1829), a French astronomer and mathematician.

and

$$\mathbf{E} = -\nabla V \tag{6.2}$$

Substituting eq. (6.2) into eq. (6.1) gives

$$\nabla \cdot (-\varepsilon \nabla V) = \rho_v \tag{6.3}$$

for an inhomogeneous medium. For a homogeneous medium, eq. (6.3) becomes

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\varepsilon}} \tag{6.4}$$

This is known as *Poisson's equation.* A special case of this equation occurs when $\rho_v = 0$ (i.e., for a charge-free region). Equation (6.4) then becomes

$$\boxed{\nabla^2 V = 0} \tag{6.5}$$

which is known as *Laplace's equation.* Note that in taking ε out of the left-hand side of eq. (6.3) to obtain eq. (6.4), we have assumed that ε is constant throughout the region in which V is defined; for an inhomogeneous region, ε is not constant and eq. (6.4) does not follow eq. (6.3). Equation (6.3) is Poisson's equation for an inhomogeneous medium; it becomes Laplace's equation for an inhomogeneous medium when $\rho_v = 0$.

Recall that the Laplacian operator ∇^2 was derived in Section 3.8. Thus Laplace's equation in Cartesian, cylindrical, or spherical coordinates respectively is given by

$$\boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0} \tag{6.6}$$

$$\boxed{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0} \tag{6.7}$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0} \tag{6.8}$$

depending on whether the potential is $V(x, y, z)$, $V(\rho, \phi, z)$, or $V(r, \theta, \phi)$. Poisson's equation in those coordinate systems may be obtained by simply replacing zero on the right-hand side of eqs. (6.6), (6.7), and (6.8) with $-\rho_v/\varepsilon$.

Laplace's equation is of primary importance in solving electrostatic problems involving a set of conductors maintained at different potentials. Examples of such problems include capacitors and vacuum tube diodes. Laplace's and Poisson's equations are not only useful in solving electrostatic field problem; they are used in various other field problems.

For example, V would be interpreted as magnetic potential in magnetostatics, as temperature in heat conduction, as stress function in fluid flow, and as pressure head in seepage.

6.3 UNIQUENESS THEOREM

Since there are several methods (analytical, graphical, numerical, experimental, etc.) of solving a given problem, we may wonder whether solving Laplace's equation in different ways gives different solutions. Therefore, before we begin to solve Laplace's equation, we should answer this question: If a solution of Laplace's equation satisfies a given set of boundary conditions, is this the only possible solution? The answer is yes: there is only one solution. We say that the solution is unique. Thus any solution of Laplace's equation which satisfies the same boundary conditions must be the only solution regardless of the method used. This is known as the *uniqueness theorem.* The theorem applies to any solution of Poisson's or Laplace's equation in a given region or closed surface.

The theorem is proved by contradiction. We assume that there are two solutions V_1 and V_2 of Laplace's equation both of which satisfy the prescribed boundary conditions. Thus

$$\nabla^2 V_1 = 0, \qquad \nabla^2 V_2 = 0 \tag{6.9a}$$

$$V_1 = V_2 \qquad \text{on the boundary} \tag{6.9b}$$

We consider their difference

$$V_d = V_2 - V_1 \tag{6.10}$$

which obeys

$$\nabla^2 V_d = \nabla^2 V_2 - \nabla^2 V_1 = 0 \tag{6.11a}$$

$$V_d = 0 \qquad \text{on the boundary} \tag{6.11b}$$

according to eq. (6.9). From the divergence theorem.

$$\int_v \nabla \cdot \mathbf{A} \, dv = \oint_S \mathbf{A} \cdot d\mathbf{S} \tag{6.12}$$

We let $\mathbf{A} = V_d \, \nabla V_d$ and use a vector identity

$$\nabla \cdot \mathbf{A} = \nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + \nabla V_d \cdot \nabla V_d$$

But $\nabla^2 V_d = 0$ according to eq. (6.11), so

$$\nabla \cdot \mathbf{A} = \nabla V_d \cdot \nabla V_d \tag{6.13}$$

Substituting eq. (6.13) into eq. (6.12) gives

$$\int_v \nabla V_d \cdot \nabla V_d \, dv = \oint_S V_d \, \nabla V_d \cdot d\mathbf{S} \tag{6.14}$$

From eqs. (6.9) and (6.11), it is evident that the right-hand side of eq. (6.14) vanishes.

Hence:

$$\int_v |\nabla V_d|^2 \, dv = 0$$

Since the integration is always positive.

$$\nabla V_d = 0 \tag{6.15a}$$

or

$$V_d = V_2 - V_1 = \text{constant everywhere in } v \tag{6.15b}$$

But eq. (6.15) must be consistent with eq. (6.9b). Hence, $V_d = 0$ or $V_1 = V_2$ everywhere, showing that V_1 and V_2 cannot be different solutions of the same problem.

This is the **uniqueness theorem:** If a solution to Laplace's equation can be found that satisfies the boundary conditions, then the solution is unique.

Similar steps can be taken to show that the theorem applies to Poisson's equation and to prove the theorem for the case where the electric field (potential gradient) is specified on the boundary.

Before we begin to solve boundary-value problems, we should bear in mind the three things that uniquely describe a problem:

1. The appropriate differential equation (Laplace's or Poisson's equation in this chapter)
2. The solution region
3. The prescribed boundary conditions

A problem does not have a unique solution and cannot be solved completely if any of the three items is missing.

6.4 GENERAL PROCEDURE FOR SOLVING POISSON'S OR LAPLACE'S EQUATION

The following general procedure may be taken in solving a given boundary-value problem involving Poisson's or Laplace's equation:

1. Solve Laplace's (if $\rho_v = 0$) or Poisson's (if $\rho_v \neq 0$) equation using either (a) direct integration when V is a function of one variable, or (b) separation of variables if V is a function of more than one variable. The solution at this point is not unique but expressed in terms of unknown integration constants to be determined.
2. Apply the boundary conditions to determine a unique solution for V. Imposing the given boundary conditions makes the solution unique.
3. Having obtained V, find $\mathbf{E}$ using $\mathbf{E} = -\nabla V$ and $\mathbf{D}$ from $\mathbf{D} = \varepsilon\mathbf{E}$.

4. If desired, find the charge Q induced on a conductor using $Q = \int \rho_S \, dS$ where $\rho_S = D_n$ and D_n is the component of $\mathbf{D}$ normal to the conductor. If necessary, the capacitance between two conductors can be found using $C = Q/V$.

Solving Laplace's (or Poisson's) equation, as in step 1, is not always as complicated as it may seem. In some cases, the solution may be obtained by mere inspection of the problem. Also a solution may be checked by going backward and finding out if it satisfies both Laplace's (or Poisson's) equation and the prescribed boundary conditions.

EXAMPLE 6.1

Current-carrying components in high-voltage power equipment must be cooled to carry away the heat caused by ohmic losses. A means of pumping is based on the force transmitted to the cooling fluid by charges in an electric field. The electrohydrodynamic (EHD) pumping is modeled in Figure 6.1. The region between the electrodes contains a uniform charge ρ_o, which is generated at the left electrode and collected at the right electrode. Calculate the pressure of the pump if $\rho_o = 25 \text{ mC/m}^3$ and $V_o = 22$ kV.

Solution:

Since $\rho_v \neq 0$, we apply Poisson's equation

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

The boundary conditions $V(z = 0) = V_o$ and $V(z = d) = 0$ show that V depends only on z (there is no ρ or ϕ dependence). Hence

$$\frac{d^2V}{dz^2} = \frac{-\rho_o}{\varepsilon}$$

Integrating once gives

$$\frac{dV}{dz} = \frac{-\rho_o z}{\varepsilon} + A$$

Integrating again yields

$$V = -\frac{\rho_o z^2}{2\varepsilon} + Az + B$$

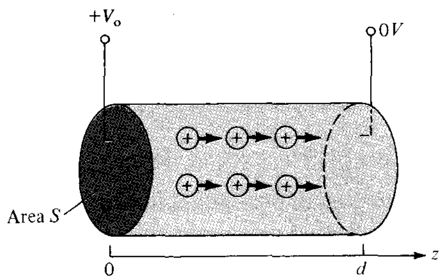


Figure 6.1 An electrohydrodynamic pump; for Example 6.1.

where A and B are integration constants to be determined by applying the boundary conditions. When $z = 0$, $V = V_o$,

$$V_o = -0 + 0 + B \rightarrow B = V_o$$

When $z = d$, $V = 0$,

$$0 = -\frac{\rho_o d^2}{2\varepsilon} + Ad + V_o$$

or

$$A = \frac{\rho_o d}{2\varepsilon} - \frac{V_o}{d}$$

The electric field is given by

$$\begin{aligned} \mathbf{E} &= -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = \left(\frac{\rho_o z}{\varepsilon} - A\right)\mathbf{a}_z \\ &= \left[\frac{V_o}{d} + \frac{\rho_o}{\varepsilon}\left(z - \frac{d}{2}\right)\right]\mathbf{a}_z \end{aligned}$$

The net force is

$$\begin{aligned} \mathbf{F} &= \int \rho_v \mathbf{E}\, dv = \rho_o \int dS \int_{z=0}^{d} \mathbf{E}\, dz \\ &= \rho_o S\left[\frac{V_o z}{d} + \frac{\rho_o}{2\varepsilon}(z^2 - dz)\right]\Bigg|_0^d \mathbf{a}_z \\ \mathbf{F} &= \rho_o S V_o \mathbf{a}_z \end{aligned}$$

The force per unit area or pressure is

$$\rho = \frac{F}{S} = \rho_o V_o = 25 \times 10^{-3} \times 22 \times 10^3 = 550 \text{ N/m}^2$$

PRACTICE EXERCISE 6.1

In a one-dimensional device, the charge density is given by $\rho_v = \rho_o x/a$. If $\mathbf{E} = 0$ at $x = 0$ and $V = 0$ at $x = a$, find V and $\mathbf{E}$.

Answer: $\dfrac{\rho_o}{6\varepsilon a}(a^3 - x^3)$, $\dfrac{\rho_o x^2}{2a\varepsilon}\mathbf{a}_x$

EXAMPLE 6.2

The xerographic copying machine is an important application of electrostatics. The surface of the photoconductor is initially charged uniformly as in Figure 6.2(a). When light from the document to be copied is focused on the photoconductor, the charges on the lower

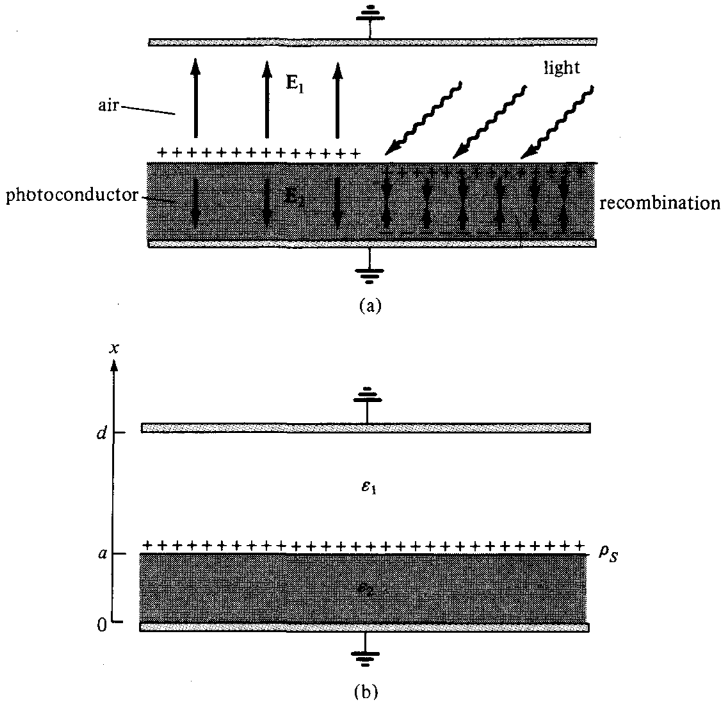


Figure 6.2 For Example 6.2.

surface combine with those on the upper surface to neutralize each other. The image is developed by pouring a charged black powder over the surface of the photoconductor. The electric field attracts the charged powder, which is later transferred to paper and melted to form a permanent image. We want to determine the electric field below and above the surface of the photoconductor.

Solution:

Consider the modeled version of Figure 6.2(a) as in Figure 6.2(b). Since $\rho_v = 0$ in this case, we apply Laplace's equation. Also the potential depends only on x. Thus

$$\nabla^2 V = \frac{d^2 V}{dx^2} = 0$$

Integrating twice gives

$$V = Ax + B$$

Let the potentials above and below be V_1 and V_2, respectively.

$$V_1 = A_1 x + B_1, \qquad x > a \tag{6.2.1a}$$

$$V_2 = A_2 x + B_2, \qquad x < a \tag{6.2.1b}$$

The boundary conditions at the grounded electrodes are

$$V_1(x = d) = 0 \tag{6.2.2.a}$$

$$V_2(x = 0) = 0 \tag{6.2.2b}$$

At the surface of the photoconductor,

$$V_1(x = a) = V_2(x = a) \tag{6.2.3a}$$

$$D_{1n} - D_{2n} = \rho_S \Big|_{x=a} \tag{6.2.3b}$$

We use the four conditions in eqs. (6.2.2) and (6.2.3) to determine the four unknown constants A_1, A_2, B_1, and B_2. From eqs. (6.2.1) and 6.2.2),

$$0 = A_1 d + B_1 \rightarrow B_1 = -A_1 d \tag{6.2.4a}$$

$$0 = 0 + B_2 \rightarrow B_2 \quad = 0 \tag{6.2.4b}$$

From eqs. (6.2.1) and (6.2.3a),

$$A_1 a + B_1 = A_2 a \tag{6.2.5}$$

To apply eq. (6.2.3b), recall that $\mathbf{D} = \varepsilon\mathbf{E} = -\varepsilon\nabla V$ so that

$$\rho_S = D_{1n} - D_{2n} = \varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = -\varepsilon_1 \frac{dV_1}{dx} + \varepsilon_2 \frac{dV_2}{dx}$$

or

$$\rho_S = -\varepsilon_1 A_1 + \varepsilon_2 A_2 \tag{6.2.6}$$

Solving for A_1 and A_2 in eqs. (6.2.4) to (6.2.6), we obtain

$$\mathbf{E}_1 = -A_1 \mathbf{a}_x = \frac{\rho_S \mathbf{a}_x}{\varepsilon_1 \left[1 + \dfrac{\varepsilon_2}{\varepsilon_1}\dfrac{d}{a} - \dfrac{\varepsilon_2}{\varepsilon_1}\right]}$$

$$\mathbf{E}_2 = -A_2 \mathbf{a}_x = \frac{-\rho_S\left(\dfrac{d}{a} - 1\right)\mathbf{a}_x}{\varepsilon_1 \left[1 + \dfrac{\varepsilon_2}{\varepsilon_1}\dfrac{d}{a} - \dfrac{\varepsilon_2}{\varepsilon_1}\right]}$$

PRACTICE EXERCISE 6.2

For the model of Figure 6.2(b), if $\rho_S = 0$ and the upper electrode is maintained at V_o while the lower electrode is grounded, show that

$$\mathbf{E}_1 = \frac{-V_o\,\mathbf{a}_x}{d - a + \dfrac{\varepsilon_1}{\varepsilon_2} a}, \qquad \mathbf{E}_2 = \frac{-V_o\,\mathbf{a}_x}{a + \dfrac{\varepsilon_2}{\varepsilon_1} d - \dfrac{\varepsilon_2}{\varepsilon_1} a}$$

EXAMPLE 6.3

Semiinfinite conducting planes $\phi = 0$ and $\phi = \pi/6$ are separated by an infinitesimal insulating gap as in Figure 6.3. If $V(\phi = 0) = 0$ and $V(\phi = \pi/6) = 100$ V, calculate V and $\mathbf{E}$ in the region between the planes.

Solution:

As V depends only on ϕ, Laplace's equation in cylindrical coordinates becomes

$$\nabla^2 V = \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0$$

Since $\rho = 0$ is excluded due to the insulating gap, we can multiply by ρ^2 to obtain

$$\frac{d^2V}{d\phi^2} = 0$$

which is integrated twice to give

$$V = A\phi + B$$

We apply the boundary conditions to determine constants A and B. When $\phi = 0$, $V = 0$,

$$0 = 0 + B \rightarrow B = 0$$

When $\phi = \phi_o$, $V = V_o$,

$$V_o = A\phi_o \rightarrow A = \frac{V_o}{\phi_o}$$

Hence:

$$V = \frac{V_o}{\phi_o}\phi$$

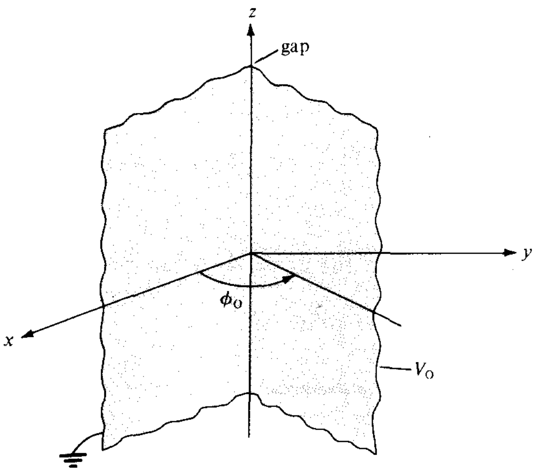


Figure 6.3 Potential $V(\phi)$ due to semi-infinite conducting planes.

and

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{V_o}{\rho\phi_o}\mathbf{a}_\phi$$

Substituting $V_o = 100$ and $\phi_o = \pi/6$ gives

$$V = \frac{600}{\pi}\phi \quad \text{and} \quad \mathbf{E} = \frac{600}{\pi\rho}\mathbf{a}_\phi$$

Check: $\nabla^2 V = 0$, $V(\phi = 0) = 0$, $V(\phi = \pi/6) = 100$.

PRACTICE EXERCISE 6.3

Two conducting plates of size 1×5 m are inclined at 45° to each other with a gap of width 4 mm separating them as shown in Figure 6.4. Determine an approximate value of the charge per plate if the plates are maintained at a potential difference of 50 V. Assume that the medium between them has $\varepsilon_r = 1.5$.

Answer: 22.2 nC.

EXAMPLE 6.4

Two conducting cones ($\theta = \pi/10$ and $\theta = \pi/6$) of infinite extent are separated by an infinitesimal gap at $r = 0$. If $V(\theta = \pi/10) = 0$ and $V(\theta = \pi/6) = 50$ V, find V and $\mathbf{E}$ between the cones.

Solution:

Consider the coaxial cone of Figure 6.5, where the gap serves as an insulator between the two conducting cones. V depends only on θ, so Laplace's equation in spherical coordinates becomes

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

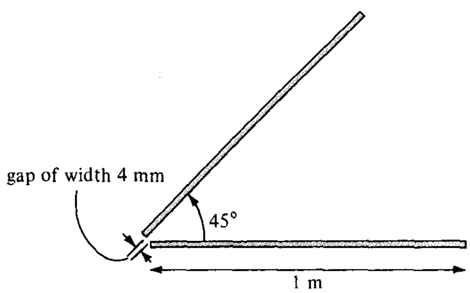


Figure 6.4 For Practice Exercise 6.3.

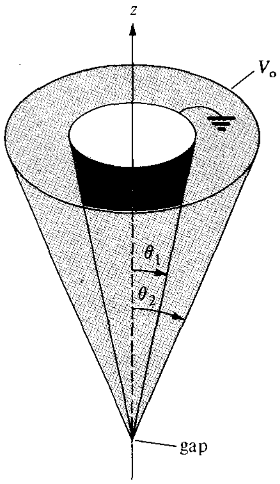


Figure 6.5 Potential $V(\phi)$ due to conducting cones.

Since $r = 0$ and $\theta = 0, \pi$ are excluded, we can multiply by $r^2 \sin\theta$ to get

$$\frac{d}{d\theta}\left[\sin\theta \frac{dV}{d\theta}\right] = 0$$

Integrating once gives

$$\sin\theta \frac{dV}{d\theta} = A$$

or

$$\frac{dV}{d\theta} = \frac{A}{\sin\theta}$$

Integrating this results in

$$\begin{aligned} V &= A \int \frac{d\theta}{\sin\theta} = A \int \frac{d\theta}{2\cos\theta/2 \sin\theta/2} \\ &= A \int \frac{1/2 \sec^2\theta/2 \, d\theta}{\tan\theta/2} \\ &= A \int \frac{d(\tan\theta/2)}{\tan\theta/2} \\ &= A \ln(\tan\theta/2) + B \end{aligned}$$

We now apply the boundary conditions to determine the integration constants A and B.

$$V(\theta = \theta_1) = 0 \rightarrow 0 = A \ln(\tan\theta_1/2) + B$$

or

$$B = -A \ln(\tan\theta_1/2)$$

Hence

$$V = A \ln \left[\frac{\tan \theta/2}{\tan \theta_1/2}\right]$$

Also

$$V(\theta = \theta_2) = V_o \rightarrow V_o = A \ln \left[\frac{\tan \theta_2/2}{\tan \theta_1/2}\right]$$

or

$$A = \frac{V_o}{\ln \left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2}\right]}$$

Thus

$$V = \frac{V_o \ln \left[\dfrac{\tan \theta/2}{\tan \theta_1/2}\right]}{\ln \left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2}\right]}$$

$$\begin{aligned}\mathbf{E} = -\nabla V &= -\frac{1}{r}\frac{dV}{d\theta}\,\mathbf{a}_\theta = -\frac{A}{r \sin \theta}\,\mathbf{a}_\theta \\ &= -\frac{V_o}{r \sin \theta \ln \left[\dfrac{\tan \theta_2/2}{\tan \theta_1/2}\right]}\,\mathbf{a}_\theta\end{aligned}$$

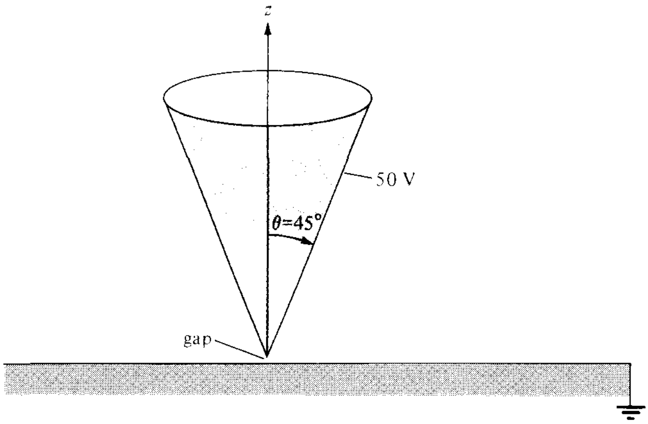
Taking $\theta_1 = \pi/10$, $\theta_2 = \pi/6$, and $V_o = 50$ gives

$$V = \frac{50 \ln \left[\dfrac{\tan \theta/2}{\tan \pi/20}\right]}{\ln \left[\dfrac{\tan \pi/12}{\tan \pi/20}\right]} = 95.1 \ln \left[\frac{\tan \theta/2}{0.1584}\right] V$$

and

$$\mathbf{E} = -\frac{95.1}{r \sin \theta}\,\mathbf{a}_\theta \text{ V/m}$$

Check: $\nabla^2 V = 0$, $V(\theta = \pi/10) = 0$, $V(\theta = \pi/6) = V_o$.



For Practice Exercise 6.4.

PRACTICE EXERCISE 6.4

A large conducting cone ($\theta = 45°$) is placed on a conducting plane with a tiny gap separating it from the plane as shown in Figure 6.6. If the cone is connected to a 50-V source, find V and $\mathbf{E}$ at $(-3, 4, 2)$.

Answer: 22.13 V, 11.36 $\mathbf{a}_\theta$ V/m.

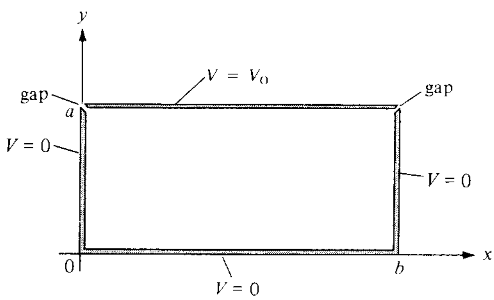
(a) Determine the potential function for the region inside the rectangular trough of infinite length whose cross section is shown in Figure 6.7.

(b) For $V_o = 100$ V and $b = 2a$, find the potential at $x = a/2$, $y = 3a/4$.

Solution:

(a) The potential V in this case depends on x and y. Laplace's equation becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \tag{6.5.1}$$



Potential $V(x, y)$ due to a conducting rectangular trough.

We have to solve this equation subject to the following boundary conditions:

$$V(x = 0, 0 \leq y \leq a) = 0 \tag{6.5.2a}$$

$$V(x = b, 0 \leq y \leq a) = 0 \tag{6.5.2b}$$

$$V(0 \leq x \leq b, y = 0) = 0 \tag{6.5.2c}$$

$$V(0 \leq x \leq b, y = a) = V_o \tag{6.5.2d}$$

We solve eq. (6.5.1) by the method of *separation of variables;* that is, we seek a product solution of V. Let

$$V(x, y) = X(x)\, Y(y) \tag{6.5.3}$$

when X is a function of x only and Y is a function of y only. Substituting eq. (6.5.3) into eq. (6.5.1) yields

$$X''Y + Y''X = 0$$

Dividing through by XY and separating X from Y gives

$$-\frac{X''}{X} = \frac{Y''}{Y} \tag{6.5.4a}$$

Since the left-hand side of this equation is a function of x only and the right-hand side is a function of y only, for the equality to hold, both sides must be equal to a constant λ; that is

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda \tag{6.5.4b}$$

The constant λ is known as the *separation constant.* From eq. (6.5.4b), we obtain

$$X'' + \lambda X = 0 \tag{6.5.5a}$$

and

$$Y'' - \lambda Y = 0 \tag{6.5.5b}$$

Thus the variables have been separated at this point and we refer to eq. (6.5.5) as *separated equations.* We can solve for *X(x)* and *Y(y)* separately and then substitute our solutions into eq. (6.5.3). To do this requires that the boundary conditions in eq. (6.5.2) be separated, if possible. We separate them as follows:

$$V(0, y) = X(0)Y(y) = 0 \rightarrow X(0) = 0 \tag{6.5.6a}$$

$$V(b, y) = X(b)Y(y) = 0 \rightarrow X(b) = 0 \tag{6.5.6b}$$

$$V(x, 0) = X(x)Y(0) = 0 \rightarrow Y(0) = 0 \tag{6.5.6c}$$

$$V(x, a) = X(0)Y(a) = V_o \text{ (inseparable)} \tag{6.5.6d}$$

To solve for $X(x)$ and $Y(y)$ in eq. (6.5.5), we impose the boundary conditions in eq. (6.5.6). We consider possible values of λ that will satisfy both the separated equations in eq. (6.5.5) and the conditions in eq. (6.5.6).

CASE A.

If $\lambda = 0$, then eq. (6.5.5a) becomes

$$X'' = 0 \qquad \text{or} \qquad \frac{d^2X}{dx^2} = 0$$

which, upon integrating twice, yields

$$X = Ax + B \tag{6.5.7}$$

The boundary conditions in eqs. (6.5.6a) and (6.5.6b) imply that

$$X(x = 0) = 0 \rightarrow 0 = 0 + B \qquad \text{or} \qquad B = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = A \cdot b + 0 \qquad \text{or} \qquad A = 0$$

because $b \neq 0$. Hence our solution for X in eq. (6.5.7) becomes

$$X(x) = 0$$

which makes $V = 0$ in eq. (6.5.3). Thus we regard $X(x) = 0$ as a trivial solution and we conclude that $\lambda \neq 0$.

CASE B.

If $\lambda < 0$, say $\lambda = -\alpha^2$, then eq. (6.5.5a) becomes

$$X'' - \alpha^2 X = 0 \qquad \text{or} \qquad (D^2 - \alpha^2)X = 0$$

where $D = \dfrac{d}{dx}$

that is,

$$DX = \pm \alpha X \tag{6.5.8}$$

showing that we have two possible solutions corresponding to the plus and minus signs. For the plus sign, eq. (6.5.8) becomes

$$\frac{dX}{dx} = \alpha X \qquad \text{or} \qquad \frac{dX}{X} = \alpha\, dx$$

Hence

$$\int \frac{dX}{X} = \int \alpha\, dx \qquad \text{or} \qquad \ln X = \alpha x + \ln A_1$$

where $\ln A_1$ is a constant of integration. Thus

$$X = A_1 e^{\alpha x} \tag{6.5.9a}$$

Similarly, for the minus sign, solving eq. (6.5.8) gives

$$X = A_2 e^{-\alpha x} \tag{6.5.9b}$$

The total solution consists of what we have in eqs. (6.5.9a) and (6.5.9b); that is,

$$X(x) = A_1 e^{\alpha x} + A_2 e^{-\alpha x} \tag{6.5.10}$$

Since $\cosh \alpha x = (e^{\alpha x} + e^{-\alpha x})/2$ and $\sinh \alpha x = (e^{\alpha x} - e^{-\alpha x})/2$ or $e^{\alpha x} = \cosh \alpha x + \sinh \alpha x$ and $e^{-\alpha x} = \cosh \alpha x - \sinh \alpha x$, eq. (6.5.10) can be written as

$$X(x) = B_1 \cosh \alpha x + B_2 \sinh \alpha x \tag{6.5.11}$$

where $B_1 = A_1 + A_2$ and $B_2 = A_1 - A_2$. In view of the given boundary conditions, we prefer eq. (6.5.11) to eq. (6.5.10) as the solution. Again, eqs. (6.5.6a) and (6.5.6b) require that

$$X(x = 0) = 0 \rightarrow 0 = B_1 \cdot (1) + B_2 \cdot (0) \qquad \text{or} \qquad B_1 = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = 0 + B_2 \sinh \alpha b$$

Since $\alpha \neq 0$ and $b \neq 0$, $\sinh \alpha b$ cannot be zero. This is due to the fact that $\sinh x = 0$ if and only if $x = 0$ as shown in Figure 6.8. Hence $B_2 = 0$ and

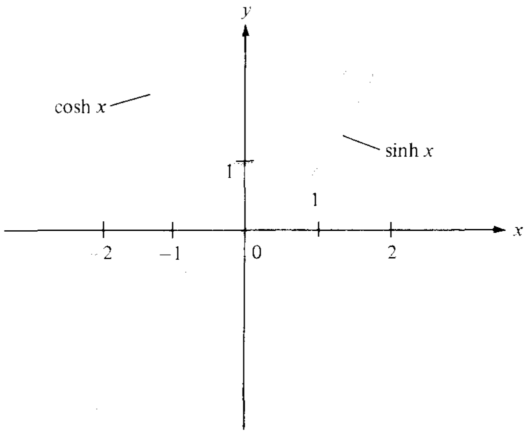
$$X(x) = 0$$

This is also a trivial solution and we conclude that λ cannot be less than zero.

CASE C.

If $\lambda > 0$, say $\lambda = \beta^2$, then eq. (6.5.5a) becomes

$$X'' + \beta^2 X = 0$$



Sketch of $\cosh x$ and $\sinh x$ showing that $\sinh x = 0$ if and only if $x = 0$.

that is,

$$(D^2 + \beta^2)X = 0 \qquad \text{or} \qquad DX = \pm j\beta X \tag{6.5.12}$$

where $j = \sqrt{-1}$. From eqs. (6.5.8) and (6.5.12), we notice that the difference between Cases 2 and 3 is replacing α by $j\beta$. By taking the same procedure as in Case 2, we obtain the solution as

$$X(x) = C_o e^{j\beta x} + C_1 e^{-j\beta x} \tag{6.5.13a}$$

Since $e^{j\beta x} = \cos \beta x + j \sin \beta x$ and $e^{-j\beta x} = \cos \beta x - j \sin \beta x$, eq. (6.5.13a) can be written as

$$X(x) = g_o \cos \beta x + g_1 \sin \beta x \tag{6.5.13b}$$

where $g_o = C_o + C_1$ and $g_1 = C_o - jC_1$.

In view of the given boundary conditions, we prefer to use eq. (6.5.13b). Imposing the conditions in eqs. (6.5.6a) and (6.5.6b) yields

$$X(x = 0) = 0 \rightarrow 0 = g_o \cdot (1) + 0 \qquad \text{or} \qquad g_o = 0$$

and

$$X(x = b) = 0 \rightarrow 0 = 0 + g_1 \sin \beta b$$

Suppose $g_1 \neq 0$ (otherwise we get a trivial solution), then

$$\sin \beta b = 0 = \sin n\pi$$

$$\beta = \frac{n\pi}{b}, \qquad n = 1, 2, 3, 4, \ldots \tag{6.5.14}$$

Note that, unlike $\sinh x$, which is zero only when $x = 0$, $\sin x$ is zero at an infinite number of points as shown in Figure 6.9. It should also be noted that $n \neq 0$ because $\beta \neq 0$; we have already considered the possibility $\beta = 0$ in Case 1 where we ended up with a trivial solution. Also we do not need to consider $n = -1, -2, -3, -4, \ldots$ because $\lambda = \beta^2$

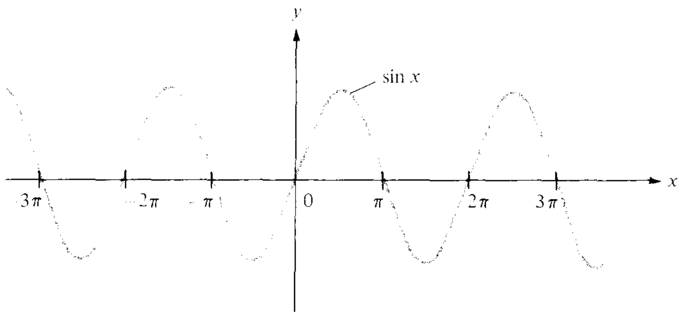


Figure 6.9 Sketch of $\sin x$ showing that $\sin x = 0$ at infinite number of points.

would remain the same for positive and negative values of n. Thus for a given n, eq. (6.5.13b) becomes

$$X_n(x) = g_n \sin \frac{n\pi x}{b} \tag{6.5.15}$$

Having found $X(x)$ and

$$\lambda = \beta^2 = \frac{n^2\pi^2}{b^2} \tag{6.5.16}$$

we solve eq. (6.5.5b) which is now

$$Y'' - \beta^2 Y = 0$$

The solution to this is similar to eq. (6.5.11) obtained in Case 2 that is,

$$Y(y) = h_o \cosh \beta y + h_1 \sinh \beta y$$

The boundary condition in eq. (6.5.6c) implies that

$$Y(y = 0) = 0 \rightarrow 0 = h_o \cdot (1) + 0 \qquad \text{or} \qquad h_o = 0$$

Hence our solution for $Y(y)$ becomes

$$Y_n(y) = h_n \sinh \frac{n\pi y}{b} \tag{6.5.17}$$

Substituting eqs. (6.5.15) and (6.5.17), which are the solutions to the separated equations in eq. (6.5.5), into the product solution in eq. (6.5.3) gives

$$V_n(x, y) = g_n h_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b}$$

This shows that there are many possible solutions V_1, V_2, V_3, V_4, and so on, for $n =$ 1, 2, 3, 4, and so on.

By the *superposition theorem,* if $V_1, V_2, V_3, \ldots, V_n$ are solutions of Laplace's equation, the linear combination

$$V = c_1V_1 + c_2V_2 + c_3V_3 + \cdots + c_nV_n$$

(where $c_1, c_2, c_3 \ldots, c_n$ are constants) is also a solution of Laplace's equation. Thus the solution to eq. (6.5.1) is

$$V(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \tag{6.5.18}$$

where $c_n = g_n h_n$ are the coefficients to be determined from the boundary condition in eq. (6.5.6d). Imposing this condition gives

$$V(x, y = a) = V_o = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi a}{b} \tag{6.5.19}$$

which is a Fourier series expansion of V_o. Multiplying both sides of eq. (6.5.19) by $\sin m\pi x/b$ and integrating over $0 < x < b$ gives

$$\int_0^b V_o \sin\frac{m\pi x}{b}\,dx = \sum_{n=1}^{\infty} c_n \sinh\frac{n\pi a}{b}\int_0^b \sin\frac{m\pi x}{b}\sin\frac{n\pi x}{b}\,dx \tag{6.5.20}$$

By the orthogonality property of the sine or cosine function (see Appendix A.9).

$$\int_0^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \end{cases}$$

Incorporating this property in eq. (6.5.20) means that all terms on the right-hand side of eq. (6.5.20) will vanish except one term in which $m = n$. Thus eq. (6.5.20) reduces to

$$\int_0^b V_o \sin\frac{n\pi x}{b}\,dx = c_n \sinh\frac{n\pi a}{b}\int_0^b \sin^2\frac{n\pi x}{b}\,dx$$

$$-V_o\frac{b}{n\pi}\cos\frac{n\pi x}{b}\bigg|_0^b = c_n \sinh\frac{n\pi a}{b}\,\frac{1}{2}\int_0^b\left(1-\cos\frac{2n\pi x}{b}\right)dx$$

$$\frac{V_o b}{n\pi}(1-\cos n\pi) = c_n \sinh\frac{n\pi a}{b}\cdot\frac{b}{2}$$

or

$$\begin{aligned} c_n \sinh\frac{n\pi a}{b} &= \frac{2V_o}{n\pi}(1-\cos n\pi) \\ &= \begin{cases} \dfrac{4V_o}{n\pi}, & n = 1, 3, 5, \ldots \\ 0, & n = 2, 4, 6, \ldots \end{cases} \end{aligned}$$

that is,

$$c_n = \begin{cases} \dfrac{4V_o}{n\pi \sinh\dfrac{n\pi a}{b}}, & n = \text{odd} \\ 0, & n = \text{even} \end{cases} \tag{6.5.21}$$

Substituting this into eq. (6.5.18) gives the complete solution as

$$V(x, y) = \frac{4V_o}{\pi}\sum_{n=1,3,5}^{\infty}\frac{\sin\dfrac{n\pi x}{b}\sinh\dfrac{n\pi y}{b}}{n\sinh\dfrac{n\pi a}{b}} \tag{6.5.22}$$

Check: $\nabla^2 V = 0$, $V(x = 0, y) = 0 = V(x = b, y) = V(x, y, = 0)$, $V(x, y = a) = V_o$. The solution in eq. (6.5.22) should not be a surprise; it can be guessed by mere observation of the potential system in Figure 6.7. From this figure, we notice that along x, V varies from

0 (at $x = 0$) to 0 (at $x = b$) and only a sine function can satisfy this requirement. Similarly, along y, V varies from 0 (at $y = 0$) to V_o (at $y = a$) and only a hyperbolic sine function can satisfy this. Thus we should expect the solution as in eq. (6.5.22).

To determine the potential for each point (x, y) in the trough, we take the first few terms of the convergent infinite series in eq. (6.5.22). Taking four or five terms may be sufficient.

(b) For $x = a/2$ and $y = 3a/4$, where $b = 2a$, we have

$$\begin{aligned} V\left(\frac{a}{2}, \frac{3a}{4}\right) &= \frac{4V_o}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\sin n\pi/4 \sinh 3n\pi/8}{n \sinh n\pi/2} \\ &= \frac{4V_o}{\pi} \left[\frac{\sin \pi/4 \sinh 3\pi/8}{\sinh \pi/2} + \frac{\sin 3\pi/4 \sinh 9\pi/8}{3 \sinh 3\pi/2} \right. \\ &\quad \left. + \frac{\sin 5\pi/4 \sinh 15\pi/4}{5 \sinh 5\pi/4} + \cdots \right] \\ &= \frac{4V_o}{\pi} (0.4517 + 0.0725 - 0.01985 - 0.00645 + 0.00229 + \cdots) \\ &= 0.6374 V_o \end{aligned}$$

It is instructive to consider a special case when $A = b = 1$ m and $V_o = 100$ V. The potentials at some specific points are calculated using eq. (6.5.22) and the result is displayed in Figure 6.10(a). The corresponding flux lines and equipotential lines are shown in Figure 6.10(b). A simple Matlab program based on eq. (6.5.22) is displayed in Figure 6.11. This self-explanatory program can be used to calculate $V(x, y)$ at any point within the trough. In Figure 6.11, $V(x = b/4, y = 3a/4)$ is typically calculated and found to be 43.2 volts.

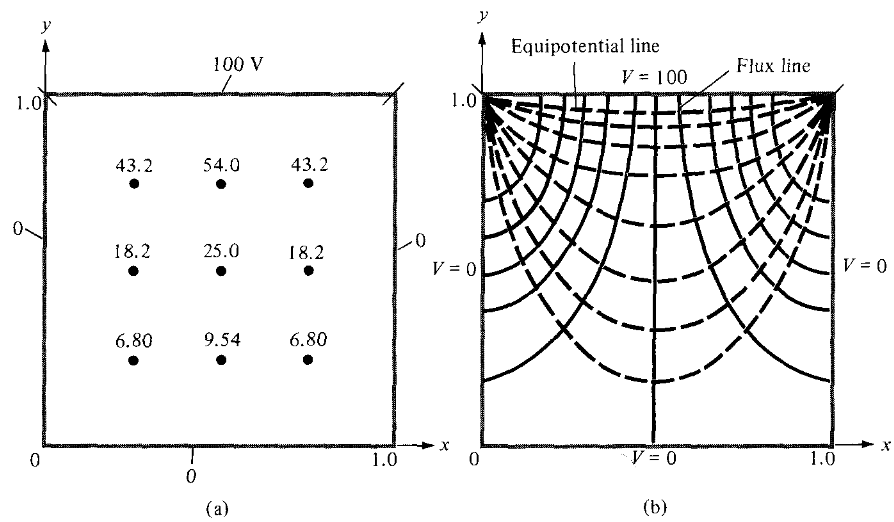


Figure 6.10 For Example 6.5: **(a)** $V(x, y)$ calculated at some points, **(b)** sketch of flux lines and equipotential lines.

```
% SOLUTION OF LAPLACE'S EQUATION
% -----------------------------
% THIS PROGRAM SOLVES THE TWO-DIMENSIONAL
% BOUNDARY-VALUE PROBLEM DESCRIBED IN FIG. 6.7
% a AND b ARE THE DIMENSIONS OF THE TROUGH
% x AND y ARE THE COORDINATES OF THE POINT
% OF INTEREST

P = [ ];
Vo = 100.0;
a = 1.0;
b = a;
x = b/4;
y= 3.*a/4.;
c = 4.*Vo/pi
sum = 0.0;
for k=1:10
   n = 2*k - 1
   a1 = sin(n*pi*x/b);
   a2 = sinh(n*pi*y/b);
   a3 = n*sinh(n*pi*a/b);
   sum = sum + c*a1*a2/a3;
   P = [n, sum]
end
diary test.out
P
diary off
```

Figure 6.11 Matlab program for Example 6.5.

PRACTICE EXERCISE 6.5

For the problem in Example 6.5, take $V_o = 100$ V, $b = 2a = 2$ m, find V and $\mathbf{E}$ at

(a) $(x, y) = (a, a/2)$

(b) $(x, y) = (3a/2, a/4)$

Answer: (a) 44.51 V, $-99.25\,\mathbf{a}_y$ V/m, (b) 16.5 V, $20.6\,\mathbf{a}_x - 70.34\,\mathbf{a}_y$ V/m.

EXAMPLE 6.6

In the last example, find the potential distribution if V_o is not constant but

(a) $V_o = 10 \sin 3\pi x/b$, $y = a$, $0 \leq x \leq b$

(b) $V_o = 2 \sin \dfrac{\pi x}{b} + \dfrac{1}{10} \sin \dfrac{5\pi x}{b}$, $y = a$, $0 \leq x \leq b$

Solution:

(a) In the last example, every step before eq. (6.5.19) remains the same; that is, the solution is of the form

$$V(x, y) = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b} \tag{6.6.1}$$

as per eq. (6.5.18). But instead of eq. (6.5.19), we now have

$$V(y = a) = V_o = 10 \sin \frac{3\pi x}{b} = \sum_{n=1}^{\infty} c_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi a}{b}$$

By equating the coefficients of the sine terms on both sides, we obtain

$$c_n = 0, \qquad n \neq 3$$

For $n = 3$,

$$10 = c_3 \sinh \frac{3\pi a}{b}$$

or

$$c_3 = \frac{10}{\sinh \dfrac{3\pi a}{b}}$$

Thus the solution in eq. (6.6.1) becomes

$$V(x, y) = 10 \sin \frac{3\pi x}{b} \frac{\sinh \dfrac{3\pi y}{b}}{\sinh \dfrac{3\pi a}{b}}$$

(b) Similarly, instead of eq. (6.5.19), we have

$$V_o = V(y = a)$$

or

$$2 \sin \frac{\pi x}{b} + \frac{1}{10} \sinh \frac{5\pi x}{b} = \sum_{n=1}^{\infty} c_n \sinh \frac{n\pi x}{b} \sinh \frac{n\pi a}{b}$$

Equating the coefficient of the sine terms:

$$c_n = 0, \qquad n \neq 1, 5$$

For $n = 1$,

$$2 = c_1 \sinh \frac{\pi a}{b} \quad \text{or} \quad c_1 = \frac{2}{\sinh \dfrac{\pi a}{b}}$$

For $n = 5$,

$$\frac{1}{10} = c_5 \sinh \frac{5\pi a}{b} \quad \text{or} \quad c_5 = \frac{1}{10 \sinh \dfrac{5\pi a}{b}}$$

Hence,

$$V(x, y) = \frac{2 \sin \dfrac{\pi x}{b} \sinh \dfrac{\pi y}{b}}{\sinh \dfrac{\pi a}{b}} + \frac{\sin \dfrac{5\pi x}{b} \sinh \dfrac{5\pi y}{b}}{10 \sinh \dfrac{5\pi a}{b}}$$

PRACTICE EXERCISE 6.6

In Example 6.5, suppose everything remains the same except that V_o is replaced by $V_o \sin \dfrac{7\pi x}{b}$, $0 \leq x \leq b$, $y = a$. Find $V(x, y)$.

Answer: $\dfrac{V_o \sin \dfrac{7\pi x}{b} \sinh \dfrac{7\pi y}{b}}{\sinh \dfrac{7\pi a}{b}}$

EXAMPLE 6.7

Obtain the separated differential equations for potential distribution $V(\rho, \phi, z)$ in a charge-free region.

Solution:

This example, like Example 6.5, further illustrates the method of separation of variables. Since the region is free of charge, we need to solve Laplace's equation in cylindrical coordinates; that is,

$$\nabla^2 V = \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{6.7.1}$$

We let

$$V(\rho, \phi, z) = R(\rho)\, \Phi(\phi)\, Z(z) \tag{6.7.2}$$

where R, Φ, and Z are, respectively, functions of ρ, ϕ, and z. Substituting eq. (6.7.2) into eq. (6.7.1) gives

$$\frac{\Phi Z}{\rho}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{RZ}{\rho^2}\frac{d^2\Phi}{d\phi^2} + R\Phi\frac{d^2Z}{dz^2} = 0 \tag{6.7.3}$$

We divide through by $R\Phi Z$ to obtain

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} \tag{6.7.4}$$

The right-hand side of this equation is solely a function of z whereas the left-hand side does not depend on z. For the two sides to be equal, they must be constant; that is,

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\frac{d^2\Phi}{d\phi^2} = -\frac{1}{Z}\frac{d^2Z}{dz^2} = -\lambda^2 \tag{6.7.5}$$

where $-\lambda^2$ is a separation constant. Equation (6.7.5) can be separated into two parts:

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = \lambda^2 \tag{6.7.6}$$

or

$$Z'' - \lambda^2 Z = 0 \tag{6.7.7}$$

and

$$\frac{\rho}{R}\frac{d}{d\rho}\left(\frac{\rho\, dR}{d\rho}\right) + \lambda^2\rho^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = 0 \tag{6.7.8}$$

Equation (6.7.8) can be written as

$$\frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + \lambda^2\rho^2 = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = \mu^2 \tag{6.7.9}$$

where μ^2 is another separation constant. Equation (6.7.9) is separated as

$$\Phi'' = \mu^2\Phi = 0 \tag{6.7.10}$$

and

$$\rho^2 R'' + \rho R' + (\rho^2\lambda^2 - \mu^2)R = 0 \tag{6.7.11}$$

Equations (6.7.7), (6.7.10), and (6.7.11) are the required separated differential equations. Equation (6.7.7) has a solution similar to the solution obtained in Case 2 of Example 6.5; that is,

$$Z(z) = c_1 \cosh \lambda z + c_2 \sinh \lambda z \tag{6.7.12}$$

The solution to eq. (6.7.10) is similar to the solution obtained in Case 3 of Example 6.5; that is,

$$\Phi(\phi) = c_3 \cos \mu\phi + c_4 \sin \mu\phi \tag{6.7.13}$$

Equation (6.7.11) is known as the *Bessel differential equation* and its solution is beyond the scope of this text.[3]

PRACTICE EXERCISE 6.7

Repeat Example 6.7 for $V(r, \theta, \phi)$.

Answer: If $V(r, \theta, \phi) = R(r)\, F(\theta)\, \Phi(\phi)$, $\Phi'' + \lambda^2\Phi = 0$, $R'' + \frac{2}{r}R' - \frac{\mu^2}{r^2}R = 0$, $F'' + \cot\theta\, F' + (\mu^2 - \lambda^2 \operatorname{cosec}^2\theta)\, F = 0$.

6.5 RESISTANCE AND CAPACITANCE

In Section 5.4 the concept of resistance was covered and we derived eq. (5.16) for finding the resistance of a conductor of uniform cross section. If the cross section of the conductor is not uniform, eq. (5.16) becomes invalid and the resistance is obtained from eq. (5.17):

$$\boxed{R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint \sigma\mathbf{E} \cdot d\mathbf{S}}} \tag{6.16}$$

The problem of finding the resistance of a conductor of nonuniform cross section can be treated as a boundary-value problem. Using eq. (6.16), the resistance R (or conductance $G = 1/R$) of a given conducting material can be found by following these steps:

1. Choose a suitable coordinate system.
2. Assume V_o as the potential difference between conductor terminals.
3. Solve Laplace's equation $\nabla^2 V$ to obtain V. Then determine $\mathbf{E}$ from $\mathbf{E} = -\nabla V$ and I from $I = \int \sigma\mathbf{E} \cdot d\mathbf{S}$.
4. Finally, obtain R as V_o/I.

In essence, we assume V_o, find I, and determine $R = V_o/I$. Alternatively, it is possible to assume current I_o, find the corresponding potential difference V, and determine R from $R = V/I_o$. As will be discussed shortly, the capacitance of a capacitor is obtained using a similar technique.

[3]For a complete solution of Laplace's equation in cylindrical or spherical coordinates, see, for example, D. T. Paris and F. K. Hurd, *Basic Electromagnetic Theory.* New York: McGraw-Hill, 1969, pp. 150–159.

Generally speaking, to have a capacitor we must have two (or more) conductors carrying equal but opposite charges. This implies that all the flux lines leaving one conductor must necessarily terminate at the surface of the other conductor. The conductors are sometimes referred to as the *plates* of the capacitor. The plates may be separated by free space or a dielectric.

Consider the two-conductor capacitor of Figure 6.12. The conductors are maintained at a potential difference V given by

$$V = V_1 - V_2 = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} \tag{6.17}$$

where **E** is the electric field existing between the conductors and conductor 1 is assumed to carry a positive charge. (Note that the **E** field is always normal to the conducting surfaces.)

We define the *capacitance* C of the capacitor as the ratio of the magnitude of the charge on one of the plates to the potential difference between them; that is,

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}}} \tag{6.18}$$

The negative sign before $V = -\int \mathbf{E} \cdot d\mathbf{l}$ has been dropped because we are interested in the absolute value of V. The capacitance C is a physical property of the capacitor and in measured in farads (F). Using eq. (6.18), C can be obtained for any given two-conductor capacitance by following either of these methods:

1. Assuming Q and determining V in terms of Q (involving Gauss's law)
2. Assuming V and determining Q in terms of V (involving solving Laplace's equation)

We shall use the former method here, and the latter method will be illustrated in Examples 6.10 and 6.11. The former method involves taking the following steps:

1. Choose a suitable coordinate system.
2. Let the two conducting plates carry charges $+Q$ and $-Q$.

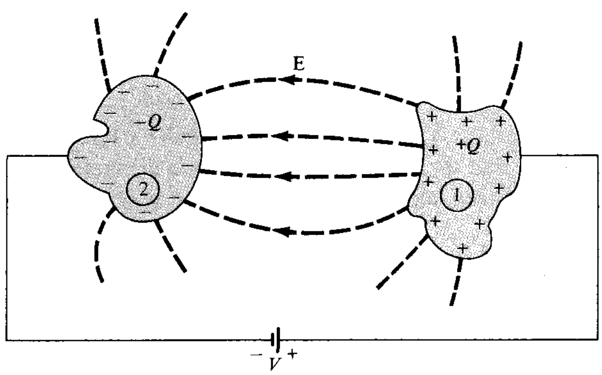


Figure 6.12 A two-conductor capacitor.

3. Determine **E** using Coulomb's or Gauss's law and find V from $V = -\int \mathbf{E} \cdot d\mathbf{l}$. The negative sign may be ignored in this case because we are interested in the absolute value of V.
4. Finally, obtain C from $C = Q/V$.

We will now apply this mathematically attractive procedure to determine the capacitance of some important two-conductor configurations.

A. Parallel-Plate Capacitor

Consider the parallel-plate capacitor of Figure 6.13(a). Suppose that each of the plates has an area S and they are separated by a distance d. We assume that plates 1 and 2, respectively, carry charges $+Q$ and $-Q$ uniformly distributed on them so that

$$\rho_S = \frac{Q}{S} \tag{6.19}$$

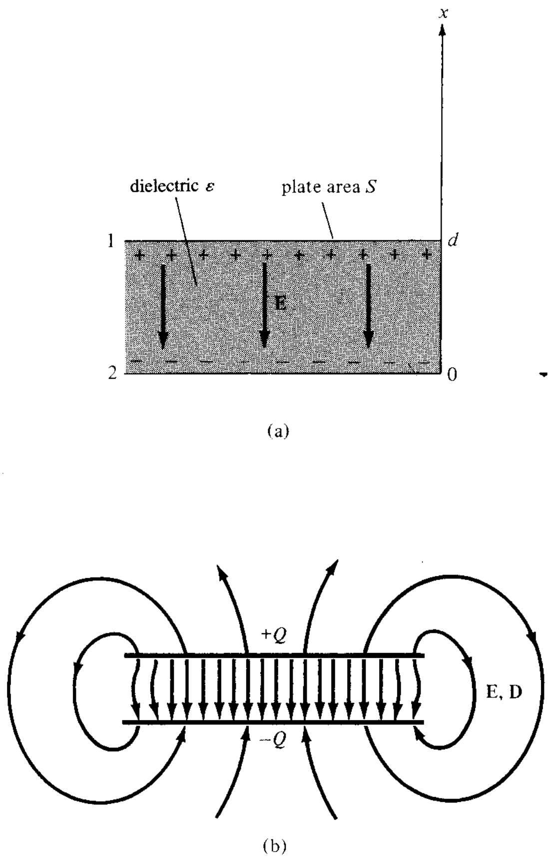


Figure 6.13 **(a)** Parallel-plate capacitor, **(b)** fringing effect due to a parallel-plate capacitor.

An ideal parallel-plate capacitor is one in which the plate separation d is very small compared with the dimensions of the plate. Assuming such an ideal case, the fringing field at the edge of the plates, as illustrated in Figure 6.13(b), can be ignored so that the field between them is considered uniform. If the space between the plates is filled with a homogeneous dielectric with permittivity ε and we ignore flux fringing at the edges of the plates, from eq. (4.27), $\mathbf{D} = -\rho_S \mathbf{a}_x$ or

$$\begin{aligned} \mathbf{E} &= \frac{\rho_S}{\varepsilon}(-\mathbf{a}_x) \\ &= -\frac{Q}{\varepsilon S}\mathbf{a}_x \end{aligned} \tag{6.20}$$

Hence

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_0^d \left[-\frac{Q}{\varepsilon S}\mathbf{a}_x\right] \cdot dx\, \mathbf{a}_x = \frac{Qd}{\varepsilon S} \tag{6.21}$$

and thus for a parallel-plate capacitor

$$\boxed{C = \frac{Q}{V} = \frac{\varepsilon S}{d}} \tag{6.22}$$

This formula offers a means of measuring the dielectric constant ε_r of a given dielectric. By measuring the capacitance C of a parallel-plate capacitor with the space between the plates filled with the dielectric and the capacitance C_o with air between the plates, we find ε_r from

$$\varepsilon_r = \frac{C}{C_o} \tag{6.23}$$

Using eq. (4.96), it can be shown that the energy stored in a capacitor is given by

$$\boxed{W_E = \frac{1}{2}CV^2 = \frac{1}{2}QV = \frac{Q^2}{2C}} \tag{6.24}$$

To verify this for a parallel-plate capacitor, we substitute eq. (6.20) into eq. (4.96) and obtain

$$\begin{aligned} W_E &= \frac{1}{2}\int_v \varepsilon \frac{Q^2}{\varepsilon^2 S^2}\, dv = \frac{\varepsilon Q^2 S d}{2\varepsilon^2 S^2} \\ &= \frac{Q^2}{2}\left(\frac{d}{\varepsilon S}\right) = \frac{Q^2}{2C} = \frac{1}{2}QV \end{aligned}$$

as expected.

B. Coaxial Capacitor

This is essentially a coaxial cable or coaxial cylindrical capacitor. Consider length L of two coaxial conductors of inner radius a and outer radius b ($b > a$) as shown in Figure 6.14. Let the space between the conductors be filled with a homogeneous dielectric with permittivity ε. We assume that conductors 1 and 2, respectively, carry $+Q$ and $-Q$ uniformly distributed on them. By applying Gauss's law to an arbitrary Gaussian cylindrical surface of radius ρ ($a < \rho < b$), we obtain

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_\rho 2\pi\rho L \tag{6.25}$$

Hence:

$$\mathbf{E} = \frac{Q}{2\pi\varepsilon\rho L}\,\mathbf{a}_\rho \tag{6.26}$$

Neglecting flux fringing at the cylinder ends,

$$V = -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{2\pi\varepsilon\rho L}\,\mathbf{a}_\rho\right] \cdot d\rho\, \mathbf{a}_\rho \tag{6.27a}$$

$$= \frac{Q}{2\pi\varepsilon L} \ln \frac{b}{a} \tag{6.27b}$$

Thus the capacitance of a coaxial cylinder is given by

$$\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\ln \dfrac{b}{a}}} \tag{6.28}$$

C. Spherical Capacitor

This is the case of two concentric spherical conductors. Consider the inner sphere of radius a and outer sphere of radius b ($b > a$) separated by a dielectric medium with permittivity ε as shown in Figure 6.15. We assume charges $+Q$ and $-Q$ on the inner and outer spheres

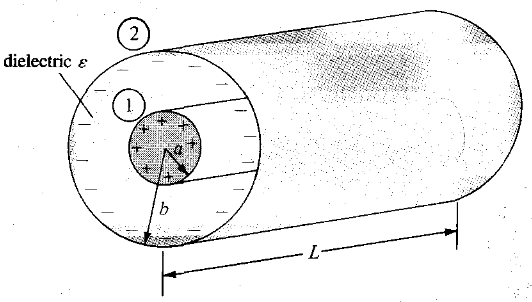


Figure 6.14 Coaxial capacitor.

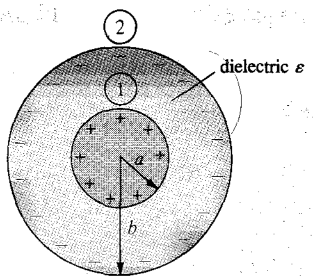


Figure 6.15 Spherical capacitor.

respectively. By applying Gauss's law to an arbitrary Gaussian spherical surface of radius $r\,(a < r < b)$,

$$Q = \varepsilon \oint \mathbf{E} \cdot d\mathbf{S} = \varepsilon E_r 4\pi r^2 \tag{6.29}$$

that is,

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r \tag{6.30}$$

The potential difference between the conductors is

$$\begin{aligned} V &= -\int_2^1 \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \left[\frac{Q}{4\pi\varepsilon r^2}\,\mathbf{a}_r\right] \cdot dr\,\mathbf{a}_r \\ &= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{a} - \frac{1}{b}\right] \end{aligned} \tag{6.31}$$

Thus the capacitance of the spherical capacitor is

$$\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}} \tag{6.32}$$

By letting $b \to \infty$, $C = 4\pi\varepsilon a$, which is the capacitance of a spherical capacitor whose outer plate is infinitely large. Such is the case of a spherical conductor at a large distance from other conducting bodies—the *isolated sphere.* Even an irregularly shaped object of about the same size as the sphere will have nearly the same capacitance. This fact is useful in estimating the stray capacitance of an isolated body or piece of equipment.

Recall from network theory that if two capacitors with capacitance C_1 and C_2 are in series (i.e., they have the same charge on them) as shown in Figure 6.16(a), the total capacitance is

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

or

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{6.33}$$

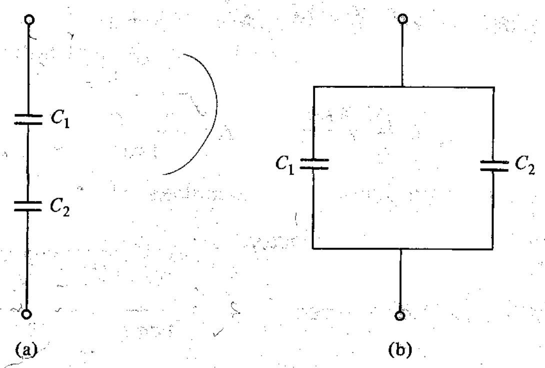


Figure 6.16 Capacitors in **(a)** series, and **(b)** parallel.

If the capacitors are in parallel (i.e., they have the same voltage across their plates) as shown in Figure 6.16(b), the total capacitance is

$$C = C_1 + C_2 \tag{6.34}$$

Let us reconsider the expressions for finding the resistance R and the capacitance C of an electrical system. The expressions were given in eqs. (6.16) and (6.18):

$$R = \frac{V}{I} = \frac{\int \mathbf{E} \cdot d\mathbf{l}}{\oint \sigma \mathbf{E} \cdot d\mathbf{S}} \tag{6.16}$$

$$C = \frac{Q}{V} = \frac{\varepsilon \oint \mathbf{E} \cdot d\mathbf{S}}{\int \mathbf{E} \cdot d\mathbf{l}} \tag{6.18}$$

The product of these expressions yields

$$\boxed{RC = \frac{\varepsilon}{\sigma}} \tag{6.35}$$

which is the relaxation time T_r of the medium separating the conductors. It should be remarked that eq. (6.35) is valid only when the medium is homogeneous; this is easily inferred from eqs. (6.16) and (6.18). Assuming homogeneous media, the resistance of various capacitors mentioned earlier can be readily obtained using eq. (6.35). The following examples are provided to illustrate this idea.

For a parallel-plate capacitor,

$$C = \frac{\varepsilon S}{d}, \qquad R = \frac{d}{\sigma S} \tag{6.36}$$

For a cylindrical capacitor,

$$C = \frac{2\pi\varepsilon L}{\ln \dfrac{b}{a}}, \qquad R = \frac{\ln \dfrac{b}{a}}{2\pi\sigma L} \tag{6.37}$$

For a spherical capacitor,

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}, \qquad R = \frac{\frac{1}{a} - \frac{1}{b}}{4\pi\sigma} \tag{6.38}$$

And finally for an isolated spherical conductor,

$$C = 4\pi\varepsilon a, \qquad R = \frac{1}{4\pi\sigma a} \tag{6.39}$$

It should be noted that the resistance R in each of eqs. (6.35) to (6.39) is not the resistance of the capacitor plate but the leakage resistance between the plates; therefore, σ in those equations is the conductivity of the dielectric medium separating the plates.

EXAMPLE 6.8

A metal bar of conductivity σ is bent to form a flat 90° sector of inner radius a, outer radius b, and thickness t as shown in Figure 6.17. Show that (a) the resistance of the bar between the vertical curved surfaces at $\rho = a$ and $\rho = b$ is

$$R = \frac{2 \ln \frac{b}{a}}{\sigma \pi t}$$

and (b) the resistance between the two horizontal surfaces at $z = 0$ and $z = t$ is

$$R' = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Solution:

(a) Between the vertical curved ends located at $\rho = a$ and $\rho = b$, the bar has a nonuniform cross section and hence eq. (5.16) does not apply. We have to use eq. (6.16). Let a potential difference V_o be maintained between the curved surfaces at $\rho = a$ and $\rho = b$ so that

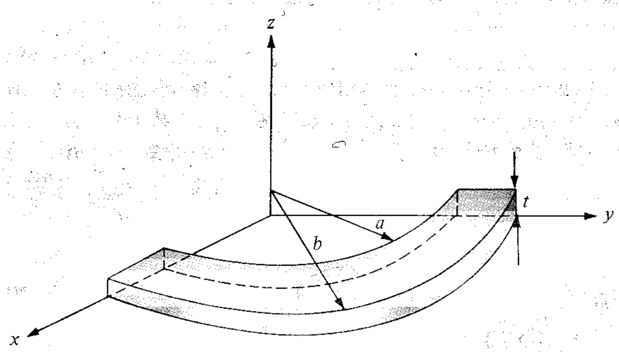


Figure 6.17 Metal bar of Example 6.8.

$V(\rho = a) = 0$ and $V(\rho = b) = V_o$. We solve for V in Laplace's equation $\nabla^2 V = 0$ in cylindrical coordinates. Since $V = V(\rho)$,

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$$

As $\rho = 0$ is excluded, upon multiplying by ρ and integrating once, this becomes

$$\rho\frac{dV}{d\rho} = A$$

or

$$\frac{dV}{d\rho} = \frac{A}{\rho}$$

Integrating once again yields

$$V = A\ln\rho + B$$

where A and B are constants of integration to be determined from the boundary conditions.

$$V(\rho = a) = 0 \to 0 = A\ln a + B \qquad \text{or} \qquad B = -A\ln a$$

$$V(\rho = b) = V_o \to V_o = A\ln b + B = A\ln b - A\ln a = A\ln\frac{b}{a} \qquad \text{or} \qquad A = \frac{V_o}{\ln\dfrac{b}{a}}$$

Hence,

$$V = A\ln\rho - A\ln a = A\ln\frac{\rho}{a} = \frac{V_o}{\ln\dfrac{b}{a}}\ln\frac{\rho}{a}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{d\rho}\mathbf{a}_\rho = -\frac{A}{\rho}\mathbf{a}_\rho = -\frac{V_o}{\rho\ln\dfrac{b}{a}}\mathbf{a}_\rho$$

$$\mathbf{J} = \sigma\mathbf{E}, \qquad d\mathbf{S} = -\rho\, d\phi\, dz\, \mathbf{a}_\rho$$

$$I = \int \mathbf{J}\cdot d\mathbf{S} = \int_{\phi=0}^{\pi/2}\int_{z=0}^{t}\frac{V_o\sigma}{\rho\ln\dfrac{b}{a}}\,dz\,\rho\,d\phi = \frac{\pi}{2}\frac{tV_o\sigma}{\ln\dfrac{b}{a}}$$

Thus

$$R = \frac{V_o}{I} = \frac{2\ln\dfrac{b}{a}}{\sigma\pi t}$$

as required.

(b) Let V_o be the potential difference between the two horizontal surfaces so that $V(z = 0) = 0$ and $V(z = t) = V_o$. $V = V(z)$, so Laplace's equation $\nabla^2 V = 0$ becomes

$$\frac{d^2V}{dz^2} = 0$$

Integrating twice gives

$$V = Az + B$$

We apply the boundary conditions to determine A and B:

$$V(z = 0) = 0 \rightarrow 0 = 0 + B \qquad \text{or} \qquad B = 0$$

$$V(z = t) = V_o \rightarrow V_o = At \qquad \text{or} \qquad A = \frac{V_o}{t}$$

Hence,

$$V = \frac{V_o}{t} z$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dz}\mathbf{a}_z = -\frac{V_o}{t}\mathbf{a}_z$$

$$\mathbf{J} = \sigma\mathbf{E} = -\frac{\sigma V_o}{t}\mathbf{a}_z, \qquad d\mathbf{S} = -\rho \, d\phi \, d\rho \, \mathbf{a}_z$$

$$I = \int \mathbf{J} \cdot d\mathbf{S} = \int_{\rho=a}^{b} \int_{\phi=0}^{\pi/2} \frac{V_o\sigma}{t} \rho \, d\phi \, d\rho$$

$$= \frac{V_o\sigma}{t} \cdot \frac{\pi}{2} \frac{\rho^2}{2}\bigg|_a^b = \frac{V_o \, \sigma \, \pi \, (b^2 - a^2)}{4t}$$

Thus

$$R' = \frac{V_o}{I} = \frac{4t}{\sigma\pi(b^2 - a^2)}$$

Alternatively, for this case, the cross section of the bar is uniform between the horizontal surfaces at $z = 0$ and $z = t$ and eq. (5.16) holds. Hence,

$$R' = \frac{\ell}{\sigma S} = \frac{t}{\sigma \frac{\pi}{4}(b^2 - a^2)}$$

$$= \frac{4t}{\sigma\pi(b^2 - a^2)}$$

as required.

PRACTICE EXERCISE 6.8

A disc of thickness t has radius b and a central hole of radius a. Taking the conductivity of the disc as σ, find the resistance between

(a) The hole and the rim of the disc

(b) The two flat sides of the disc

Answer: (a) $\dfrac{\ln b/a}{2\pi t\sigma}$, (b) $\dfrac{t}{\sigma\pi(b^2 - a^2)}$.

A coaxial cable contains an insulating material of conductivity σ. If the radius of the central wire is a and that of the sheath is b, show that the conductance of the cable per unit length is (see eq. (6.37))

$$G = \frac{2\pi\sigma}{\ln b/a}$$

Solution:

Consider length L of the coaxial cable as shown in Figure 6.14. Let V_o be the potential difference between the inner and outer conductors so that $V(\rho = a) = 0$ and $V(\rho = b) = V_o$ V and $\mathbf{E}$ can be found just as in part (a) of the last example. Hence:

$$\mathbf{J} = \sigma\mathbf{E} = \frac{-\sigma V_o}{\rho \ln b/a}\,\mathbf{a}_\rho, \qquad d\mathbf{S} = -\rho d\phi\, dz\, \mathbf{a}_\rho$$

$$\begin{aligned} I &= \int \mathbf{J} \cdot d\mathbf{S} = \int_{\phi=0}^{2\pi} \int_{z=0}^{L} \frac{V_o\sigma}{\rho \ln b/a}\,\rho\, dz\, d\phi \\ &= \frac{2\pi L\sigma V_o}{\ln b/a} \end{aligned}$$

The resistance per unit length is

$$R = \frac{V_o}{I} \cdot \frac{1}{L} = \frac{\ln b/a}{2\pi\sigma}$$

and the conductance per unit length is

$$G = \frac{1}{R} = \frac{2\pi\sigma}{\ln b/a}$$

as required.

PRACTICE EXERCISE 6.9

A coaxial cable contains an insulating material of conductivity σ_1 in its upper half and another material of conductivity σ_2 in its lower half (similar to the situation in Figure 6.19b). If the radius of the central wire is a and that of the sheath is b, show that the leakage resistance of length ℓ of the cable is

$$R = \frac{1}{\pi\ell(\sigma_1 + \sigma_2)} \ln \frac{b}{a}$$

Answer: Proof.

EXAMPLE 6.10

Conducting spherical shells with radii a = 10 cm and b = 30 cm are maintained at a potential difference of 100 V such that $V(r = b) = 0$ and $V(r = a) = 100$ V. Determine V and **E** in the region between the shells. If $\varepsilon_r = 2.5$ in the region, determine the total charge induced on the shells and the capacitance of the capacitor.

Solution:

Consider the spherical shells shown in Figure 6.18. V depends only on r and hence Laplace's equation becomes

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

Since $r \neq 0$ in the region of interest, we multiply through by r^2 to obtain

$$\frac{d}{dr}\left[r^2 \frac{dV}{dr}\right] = 0$$

Integrating once gives

$$r^2 \frac{dV}{dr} = A$$

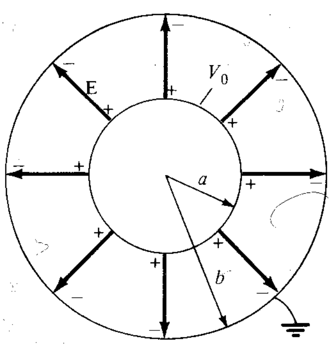


Figure 6.18 Potential $V(r)$ due to conducting spherical shells.

or

$$\frac{dV}{dr} = \frac{A}{r^2}$$

Integrating again gives

$$V = -\frac{A}{r} + B$$

As usual, constants A and B are determined from the boundary conditions.

When $r = b, V = 0 \rightarrow 0 = -\frac{A}{b} + B \quad \text{or} \quad B = \frac{A}{b}$

Hence

$$V = A\left[\frac{1}{b} - \frac{1}{r}\right]$$

Also when $r = a, V = V_o \rightarrow V_o = A\left[\frac{1}{b} - \frac{1}{a}\right]$

or

$$A = \frac{V_o}{\dfrac{1}{b} - \dfrac{1}{a}}$$

Thus

$$V = V_o \frac{\left[\dfrac{1}{r} - \dfrac{1}{b}\right]}{\dfrac{1}{a} - \dfrac{1}{b}}$$

$$\mathbf{E} = -\nabla V = -\frac{dV}{dr}\,\mathbf{a}_r = -\frac{A}{r^2}\,\mathbf{a}_r$$

$$= \frac{V_o}{r^2\left[\dfrac{1}{a} - \dfrac{1}{b}\right]}\,\mathbf{a}_r$$

$$Q = \int \varepsilon \mathbf{E} \cdot d\mathbf{S} = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\varepsilon_o \varepsilon_r V_o}{r^2\left[\dfrac{1}{a} - \dfrac{1}{b}\right]} r^2 \sin\theta \, d\phi \, d\theta$$

$$= \frac{4\pi\varepsilon_o\varepsilon_r V_o}{\dfrac{1}{a} - \dfrac{1}{b}}$$

The capacitance is easily determined as

$$C = \frac{Q}{V_o} = \frac{4\pi\varepsilon}{\dfrac{1}{a} - \dfrac{1}{b}}$$

which is the same as we obtained in eq. (6.32); there in Section 6.5, we assumed Q and found the corresponding V_o, but here we assumed V_o and found the corresponding Q to determine C. Substituting $a = 0.1$ m, $b = 0.3$ m, $V_o = 100$ V yields

$$V = 100\frac{\left[\dfrac{1}{r} - \dfrac{10}{3}\right]}{10 - 10/3} = 15\left[\frac{1}{r} - \frac{10}{3}\right] V$$

Check: $\nabla^2 V = 0$, $V(r = 0.3\text{ m}) = 0$, $V(r = 0.1\text{ m}) = 100$.

$$\mathbf{E} = \frac{100}{r^2\,[10 - 10/3]}\,\mathbf{a}_r = \frac{15}{r^2}\,\mathbf{a}_r \text{ V/m}$$

$$Q = \pm 4\pi \cdot \frac{10^{-9}}{36\pi} \cdot \frac{(2.5)\cdot(100)}{10 - 10/3}$$
$$= \pm 4.167 \text{ nC}$$

The positive charge is induced on the inner shell; the negative charge is induced on the outer shell. Also

$$C = \frac{|Q|}{V_o} = \frac{4.167 \times 10^{-9}}{100} = 41.67 \text{ pF}$$

PRACTICE EXERCISE 6.10

If Figure 6.19 represents the cross sections of two spherical capacitors, determine their capacitances. Let $a = 1$ mm, $b = 3$ mm, $c = 2$ mm, $\varepsilon_{r1} = 2.5$, and $\varepsilon_{r2} = 3.5$.

Answer: (a) 0.53 pF, (b) 0.5 pF

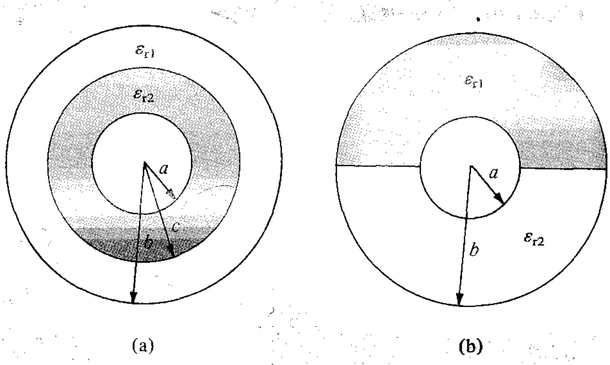


Figure 6.19 For Practice Exercises 6.9, 6.10, and 6.12.

EXAMPLE 6.11

In Section 6.5, it was mentioned that the capacitance $C = Q/V$ of a capacitor can be found by either assuming Q and finding V or by assuming V and finding Q. The former approach was used in Section 6.5 while we have used the latter method in the last example. Using the latter method, derive eq. (6.22).

Solution:

Assume that the parallel plates in Figure 6.13 are maintained at a potential difference V_o so that $V(x = 0)$ and $V(x = d) = V_o$. This necessitates solving a one-dimensional boundary-value problem; that is, we solve Laplace's equation

$$\nabla^2 V = \frac{d^2V}{dx^2} = 0$$

Integrating twice gives

$$V = Ax + B$$

where A and B are integration constants to be determined from the boundary conditions. At $x = 0$, $V = 0 \rightarrow 0 = 0 + B$, or $B = 0$, and at $x = d$, $V = V_o \rightarrow V_o = Ad + 0$ or $A = V_o/d$.

Hence

$$V = \frac{V_o}{d} x$$

Notice that this solution satisfies Laplace's equation and the boundary conditions.

We have assumed the potential difference between the plates to be V_o. Our goal is to find the charge Q on either plate so that we can eventually find the capacitance $C = Q/V_o$. The charge on either plate is

$$Q = \int \rho_S \, dS$$

But $\rho_S = \mathbf{D} \cdot \mathbf{a}_n = \varepsilon \mathbf{E} \cdot \mathbf{a}_n$, where

$$\mathbf{E} = -\nabla V = -\frac{dV}{dx}\mathbf{a}_x = -A\mathbf{a}_x = -\frac{V_o}{d}\mathbf{a}_x$$

On the lower plates, $\mathbf{a}_n = \mathbf{a}_x$, so

$$\rho_S = -\frac{\varepsilon V_o}{d} \quad \text{and} \quad Q = -\frac{\varepsilon V_o S}{d}$$

On the upper plates, $\mathbf{a}_n = -\mathbf{a}_x$, so

$$\rho_S = \frac{\varepsilon V_o}{d} \quad \text{and} \quad Q = \frac{\varepsilon V_o S}{d}$$

As expected, Q is equal but opposite on each plate. Thus

$$C = \frac{|Q|}{V_o} = \frac{\varepsilon S}{d}$$

which is in agreement with eq. (6.22).

PRACTICE EXERCISE 6.11

Derive the formula for the capacitance $C = Q/V_o$ of a cylindrical capacitor in eq. (6.28) by assuming V_o and finding Q.

EXAMPLE 6.12

Determine the capacitance of each of the capacitors in Figure 6.20. Take $\varepsilon_{r1} = 4$, $\varepsilon_{r2} = 6$, $d = 5$ mm, $S = 30$ cm^2.

Solution:

(a) Since **D** and **E** are normal to the dielectric interface, the capacitor in Figure 6.20(a) can be treated as consisting of two capacitors C_1 and C_2 in series as in Figure 6.16(a).

$$C_1 = \frac{\varepsilon_o \varepsilon_{r1} S}{d/2} = \frac{2\varepsilon_o \varepsilon_{r1} S}{d}, \qquad C_2 = \frac{2\varepsilon_o \varepsilon_{r2} S}{d}$$

The total capacitor C is given by

$$\begin{aligned} C &= \frac{C_1 C_2}{C_1 + C_2} = \frac{2\varepsilon_o S}{d} \frac{(\varepsilon_{r1}\varepsilon_{r2})}{\varepsilon_{r1} + \varepsilon_{r2}} \\ &= 2 \cdot \frac{10^{-9}}{36\pi} \cdot \frac{30 \times 10^{-4}}{5 \times 10^{-3}} \cdot \frac{4 \times 6}{10} \qquad (6.12.1) \\ C &= 25.46 \text{ pF} \end{aligned}$$

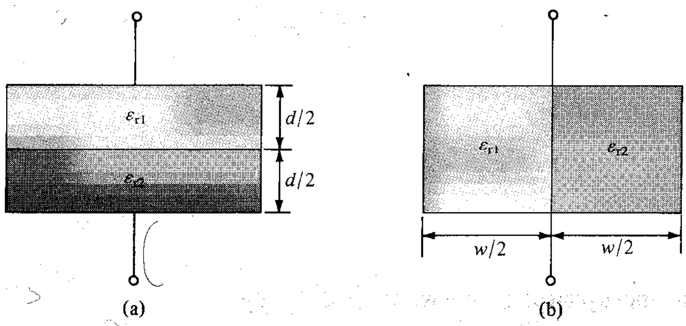


Figure 6.20 For Example 6.12.

(b) In this case, **D** and **E** are parallel to the dielectric interface. We may treat the capacitor as consisting of two capacitors C_1 and C_2 in parallel (the same voltage across C_1 and C_2) as in Figure 6.16(b).

$$C_1 = \frac{\varepsilon_o \varepsilon_{r1} S/2}{d} = \frac{\varepsilon_o \varepsilon_{r1} S}{2d}, \qquad C_2 = \frac{\varepsilon_o \varepsilon_{r2} S}{2d}$$

The total capacitance is

$$\begin{aligned} C &= C_1 + C_2 = \frac{\varepsilon_o S}{2d}(\varepsilon_{r1} + \varepsilon_{r2}) \\ &= \frac{10^{-9}}{36\pi} \cdot \frac{30 \times 10^{-4}}{2 \cdot (5 \times 10^{-3})} \cdot 10 \\ C &= 26.53 \text{ pF} \end{aligned} \tag{6.12.2}$$

Notice that when $\varepsilon_{r1} = \varepsilon_{r2} = \varepsilon_r$, eqs. (6.12.1) and (6.12.2) agree with eq. (6.22) as expected.

PRACTICE EXERCISE 6.12

Determine the capacitance of 10 m length of the cylindrical capacitors shown in Figure 6.19. Take $a = 1$ mm, $b = 3$ mm, $c = 2$ mm, $\varepsilon_{r1} = 2.5$, and $\varepsilon_{r2} = 3.5$.

Answer: (a) 1.41 nF, (b) 1.52 nF.

A cylindrical capacitor has radii $a = 1$ cm and $b = 2.5$ cm. If the space between the plates is filled with an inhomogeneous dielectric with $\varepsilon_r = (10 + \rho)/\rho$, where ρ is in centimeters, find the capacitance per meter of the capacitor.

Solution:

The procedure is the same as that taken in Section 6.5 except that eq. (6.27a) now becomes

$$\begin{aligned} V &= -\int_b^a \frac{Q}{2\pi\varepsilon_o\varepsilon_r\rho L}\,d\rho = -\frac{Q}{2\pi\varepsilon_o L}\int_b^a \frac{d\rho}{\rho\left(\dfrac{10+\rho}{\rho}\right)} \\ &= \frac{-Q}{2\pi\varepsilon_o L}\int_b^a \frac{d\rho}{10+\rho} = \frac{-Q}{2\pi\varepsilon_o L}\ln(10+\rho)\Bigg|_b^a \\ &= \frac{Q}{2\pi\varepsilon_o L}\ln\frac{10+b}{10+a} \end{aligned}$$

Thus the capacitance per meter is ($L = 1$ m)

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_o}{\ln\dfrac{10+b}{10+a}} = 2\pi \cdot \frac{10^{-9}}{36\pi} \cdot \frac{1}{\ln\dfrac{12.5}{11.0}}$$

$$C = 434.6 \text{ pF/m}$$

PRACTICE EXERCISE 6.13

A spherical capacitor with $a = 1.5$ cm, $b = 4$ cm has an inhomogeneous dielectric of $\varepsilon = 10\varepsilon_o/r$. Calculate the capacitance of the capacitor.

Answer: 1.13 nF.

6.6 METHOD OF IMAGES

The method of images, introduced by Lord Kelvin in 1848, is commonly used to determine V, $\mathbf{E}$, $\mathbf{D}$, and ρ_S due to charges in the presence of conductors. By this method, we avoid solving Poisson's or Laplace's equation but rather utilize the fact that a conducting surface is an equipotential. Although the method does not apply to all electrostatic problems, it can reduce a formidable problem to a simple one.

theory states that a given charge configuration above an infinite conducting plane may be replaced by the charge configuration ... n equipotential surface in place of the conducting plane.

Typical examples of point, line, and volume charge configurations are portrayed in Figure 6.21(a), and their corresponding image configurations are in Figure 6.21(b).

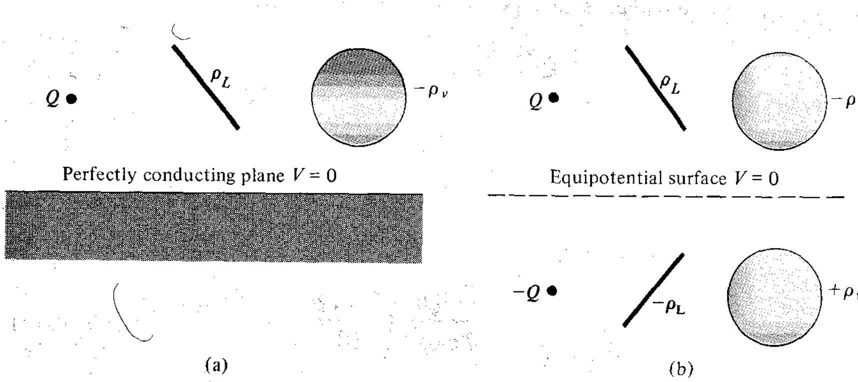


Figure 6.21 Image system: **(a)** charge configurations above a perfectly conducting plane; **(b)** image configuration with the conducting plane replaced by equipotential surface.

In applying the image method, two conditions must always be satisfied:

1. The image charge(s) must be located in the conducting region.
2. The image charge(s) must be located such that on the conducting surface(s) the potential is zero or constant.

The first condition is necessary to satisfy Poisson's equation, and the second condition ensures that the boundary conditions are satisfied. Let us now apply the image theory to two specific problems.

A. A Point Charge Above a Grounded Conducting Plane

Consider a point charge Q placed at a distance h from a perfect conducting plane of infinite extent as in Figure 6.22(a). The image configuration is in Figure 6.22(b). The electric field at point $P(x, y, z)$ is given by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{6.40}$$

$$= \frac{Q\,\mathbf{r}_1}{4\pi\varepsilon_o r_1^3} + \frac{-Q\,\mathbf{r}_2}{4\pi\varepsilon_o r_2^3} \tag{6.41}$$

The distance vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are given by

$$\mathbf{r}_1 = (x, y, z) - (0, 0, h) = (x, y, z - h) \tag{6.42}$$

$$\mathbf{r}_2 = (x, y, z) - (0, 0, -h) = (x, y, z + h) \tag{6.43}$$

so eq. (6.41) becomes

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + y\mathbf{a}_y + (z - h)\mathbf{a}_z}{[x^2 + y^2 + (z - h)^2]^{3/2}} - \frac{x\mathbf{a}_x + y\mathbf{a}_y + (z + h)\mathbf{a}_z}{[x^2 + y^2 + (z + h)^2]^{3/2}}\right] \tag{6.44}$$

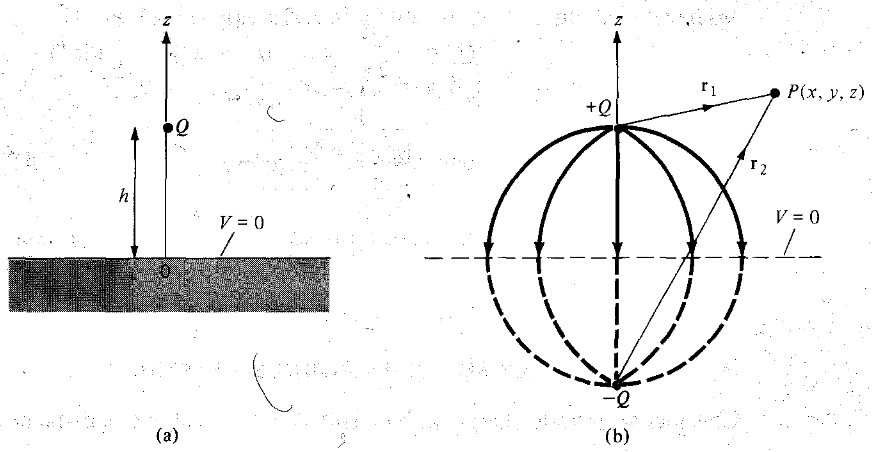


Figure 6.22 **(a)** Point charge and grounded conducting plane, **(b)** image configuration and field lines.

It should be noted that when $z = 0$, $\mathbf{E}$ has only the z-component, confirming that $\mathbf{E}$ is normal to the conducting surface.

The potential at P is easily obtained from eq. (6.41) or (6.44) using $V = -\int \mathbf{E} \cdot d\mathbf{l}$. Thus

$$\begin{aligned} V &= V_+ + V_- \\ &= \frac{Q}{4\pi\varepsilon_o r_1} + \frac{-Q}{4\pi\varepsilon_o r_2} \\ V &= \frac{Q}{4\pi\varepsilon_o}\left\{\frac{1}{[x^2 + y^2 + (z-h)^2]^{1/2}} - \frac{1}{[x^2 + y^2 + (z+h)^2]^{1/2}}\right\} \end{aligned} \tag{6.45}$$

for $z \geq 0$ and $V = 0$ for $z \leq 0$. Note that $V(z = 0) = 0$.

The surface charge density of the induced charge can also be obtained from eq. (6.44) as

$$\begin{aligned} \rho_S = D_n = \varepsilon_o E_n \Big|_{z=0} \\ = \frac{-Qh}{2\pi[x^2 + y^2 + h^2]^{3/2}} \end{aligned} \tag{6.46}$$

The total induced charge on the conducting plane is

$$Q_i = \int \rho_S \, dS = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-Qh\, dx\, dy}{2\pi[x^2 + y^2 + h^2]^{3/2}} \tag{6.47}$$

By changing variables, $\rho^2 = x^2 + y^2$, $dx\, dy = \rho\, d\rho\, d\phi$.

$$Q_i = -\frac{Qh}{2\pi}\int_0^{2\pi}\int_0^{\infty} \frac{\rho\, d\rho\, d\phi}{[\rho^2 + h^2]^{3/2}} \tag{6.48}$$

Integrating over ϕ gives 2π, and letting $\rho\, d\rho = \frac{1}{2}d\,(\rho^2)$, we obtain

$$\begin{aligned} Q_i &= -\frac{Qh}{2\pi} 2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2} \frac{1}{2} d(\rho^2) \\ &= \frac{Qh}{[\rho^2 + h^2]^{1/2}}\Bigg|_0^{\infty} \\ &= -Q \end{aligned} \tag{6.49}$$

as expected, because all flux lines terminating on the conductor would have terminated on the image charge if the conductor were absent.

B. A Line Charge above a Grounded Conducting Plane

Consider an infinite charge with density ρ_L C/m located at a distance h from the grounded conducting plane $z = 0$. The same image system of Figure 6.22(b) applies to the line charge except that Q is replaced by ρ_L. The infinite line charge ρ_L may be assumed to be at

$x = 0, z = h$ and the image $-\rho_L$ at $x = 0, z = -h$ so that the two are parallel to the y-axis. The electric field at point P is given (from eq. 4.21) by

$$\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- \tag{6.50}$$

$$= \frac{\rho_L}{2\pi\varepsilon_o\rho_1}\mathbf{a}_{\rho 1} + \frac{-\rho_L}{2\pi\varepsilon_o\rho_2}\mathbf{a}_{\rho 2} \tag{6.51}$$

The distance vectors $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ are given by

$$\boldsymbol{\rho}_1 = (x, y, z) - (0, y, h) = (x, 0, z - h) \tag{6.52}$$

$$\boldsymbol{\rho}_2 = (x, y, z) - (0, y, -h) = (x, 0, z + h) \tag{6.53}$$

so eq. (6.51) becomes

$$\mathbf{E} = \frac{\rho_L}{2\pi\varepsilon_o}\left[\frac{x\mathbf{a}_x + (z - h)\mathbf{a}_z}{x^2 + (z - h)^2} - \frac{x\mathbf{a}_x + (z + h)\mathbf{a}_z}{x^2 + (z + h)^2}\right] \tag{6.54}$$

Again, notice that when $z = 0$, $\mathbf{E}$ has only the z-component, confirming that $\mathbf{E}$ is normal to the conducting surface.

The potential at P is obtained from eq. (6.51) or (6.54) using $V = -\int \mathbf{E} \cdot d\mathbf{l}$. Thus

$$\begin{aligned} V &= V_+ + V_- \\ &= -\frac{\rho_L}{2\pi\varepsilon_o}\ln\rho_1 - \frac{-\rho_L}{2\pi\varepsilon_o}\ln\rho_2 \\ &= -\frac{\rho_L}{2\pi\varepsilon_o}\ln\frac{\rho_1}{\rho_2} \end{aligned} \tag{6.55}$$

Substituting $\rho_1 = |\boldsymbol{\rho}_1|$ and $\rho_2 = |\boldsymbol{\rho}_2|$ in eqs. (6.52) and (6.53) into eq. (6.55) gives

$$V = -\frac{\rho_L}{2\pi\varepsilon_o}\ln\left[\frac{x^2 + (z - h)^2}{x^2 + (z + h)^2}\right]^{1/2} \tag{6.56}$$

for $z \geq 0$ and $V = 0$ for $z \leq 0$. Note that $V(z = 0) = 0$.

The surface charge induced on the conducting plane is given by

$$\rho_S = D_n = \varepsilon_o E_z\Big|_{z=0} = \frac{-\rho_L h}{\pi(x^2 + h^2)} \tag{6.57}$$

The induced charge per length on the conducting plane is

$$\rho_i = \int \rho_S\, dx = -\frac{\rho_L h}{\pi}\int_{-\infty}^{\infty}\frac{dx}{x^2 + h^2} \tag{6.58}$$

By letting $x = h\tan\alpha$, eq. (6.58) becomes

$$\begin{aligned} \rho_i &= -\frac{\rho_L h}{\pi}\int_{-\pi/2}^{\pi/2}\frac{d\alpha}{h} \\ &= -\rho_L \end{aligned} \tag{6.59}$$

as expected.

EXAMPLE 6.14

A point charge Q is located at point $(a, 0, b)$ between two semiinfinite conducting planes intersecting at right angles as in Figure 6.23. Determine the potential at point $P(x, y, z)$ and the force on Q.

Solution:

The image configuration is shown in Figure 6.24. Three image charges are necessary to satisfy the conditions in Section 6.6. From Figure 6.24(a), the potential at point $P(x, y, z)$ is the superposition of the potentials at P due to the four point charges; that is,

$$V = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right]$$

where

$$r_1 = [(x - a)^2 + y^2 + (z - b)^2]^{1/2}$$
$$r_2 = [(x + a)^2 + y^2 + (z - b)^2]^{1/2}$$
$$r_3 = [(x + a)^2 + y^2 + (z + b)^2]^{1/2}$$
$$r_4 = [(x - a)^2 + y^2 + (z + b)^2]^{1/2}$$

From Figure 6.24(b), the net force on Q

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\\ &= -\frac{Q^2}{4\pi\varepsilon_o(2b)^2}\mathbf{a}_z - \frac{Q^2}{4\pi\varepsilon_o(2a)^2}\mathbf{a}_x + \frac{Q^2(2a\mathbf{a}_x + 2b\mathbf{a}_z)}{4\pi\varepsilon_o[(2a)^2 + (2b)^2]^{3/2}}\\ &= \frac{Q^2}{16\pi\varepsilon_o}\left\{\left[\frac{a}{(a^2 + b^2)^{3/2}} - \frac{1}{a^2}\right]\mathbf{a}_x + \left[\frac{b}{(a^2 + b^2)^{3/2}} - \frac{1}{b^2}\right]\mathbf{a}_z\right\}\end{aligned}$$

The electric field due to this system can be determined similarly and the charge induced on the planes can also be found.

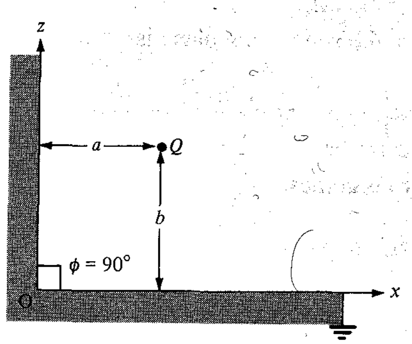


Figure 6.23 Point charge between two semiinfinite conducting planes.

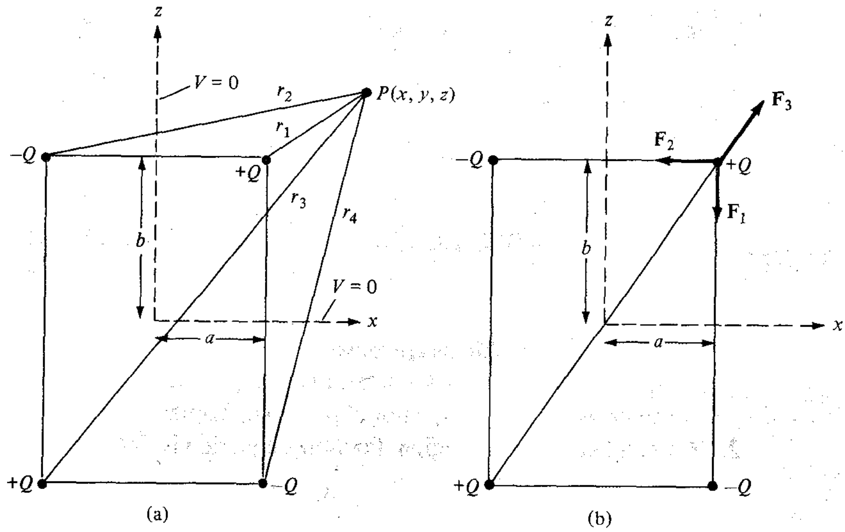


Figure 6.24 Determining **(a)** the potential at P, and **(b)** the force on charge Q.

In general, when the method of images is used for a system consisting of a point charge between two semiinfinite conducting planes inclined at an angle ϕ (in degrees), the number of images is given by

$$\boxed{N = \left(\frac{360°}{\phi} - 1\right)}$$

because the charge and its images all lie on a circle. For example, when $\phi = 180°$, $N = 1$ as in the case of Figure 6.22; for $\phi = 90°$, $N = 3$ as in the case of Figure 6.23; and for $\phi = 60°$, we expect $N = 5$ as shown in Figure 6.25.

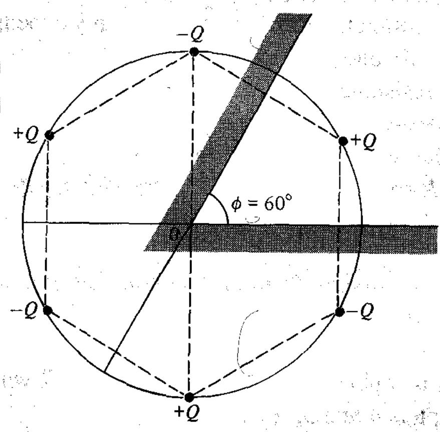


Figure 6.25 Point charge between two semiinfinite conducting walls inclined at $\phi = 60°$ to each.

PRACTICE EXERCISE 6.14

If the point charge $Q = 10$ nC in Figure 6.25 is 10 cm away from point O and along the line bisecting $\phi = 60°$, find the magnitude of the force on Q due to the charge induced on the conducting walls.

Answer: 60.53 μN.

SUMMARY

1. Boundary-value problems are those in which the potentials at the boundaries of a region are specified and we are to determine the potential field within the region. They are solved using Poisson's equation if $\rho_v \neq 0$ or Laplace's equation if $\rho_v = 0$.
2. In a nonhomogeneous region, Poisson's equation is

$$\nabla \cdot \varepsilon \nabla V = -\rho_v$$

For a homogeneous region, ε is independent of space variables. Poisson's equation becomes

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

In a charge-free region ($\rho_v = 0$), Poisson's equation becomes Laplace's equation; that is,

$$\nabla^2 V = 0$$

3. We solve the differential equation resulting from Poisson's or Laplace's equation by integrating twice if V depends on one variable or by the method of separation of variables if V is a function of more than one variable. We then apply the prescribed boundary conditions to obtain a unique solution.
4. The uniqueness theorem states that if V satisfies Poisson's or Laplace's equation and the prescribed boundary condition, V is the only possible solution for that given problem. This enables us to find the solution to a given problem via any expedient means because we are assured of one, and only one, solution.
5. The problem of finding the resistance R of an object or the capacitance C of a capacitor may be treated as a boundary-value problem. To determine R, we assume a potential difference V_o between the ends of the object, solve Laplace's equation, find $I = \int \sigma \mathbf{E} \cdot d\mathbf{S}$, and obtain $R = V_o/I$. Similarly, to determine C, we assume a potential difference of V_o between the plates of the capacitor, solve Laplace's equation, find $Q = \int \varepsilon \mathbf{E} \cdot d\mathbf{S}$, and obtain $C = Q/V_o$.
6. A boundary-value problem involving an infinite conducting plane or wedge may be solved using the method of images. This basically entails replacing the charge configuration by itself, its image, and an equipotential surface in place of the conducting plane. Thus the original problem is replaced by "an image problem," which is solved using techniques covered in Chapters 4 and 5.

REVIEW QUESTIONS

6.1 Equation $\nabla \cdot (-\varepsilon \nabla V) = \rho_v$ may be regarded as Poisson's equation for an inhomogeneous medium.

(a) True
(b) False

6.2 In cylindrical coordinates, equation

$$\frac{\partial^2 \psi}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} + 10 = 0$$

is called

(a) Maxwell's equation
(b) Laplace's equation
(c) Poisson's equation
(d) Helmholtz's equation
(e) Lorentz's equation

6.3 Two potential functions V_1 and V_2 satisfy Laplace's equation within a closed region and assume the same values on its surface. V_1 must be equal to V_2.

(a) True
(b) False
(c) Not necessarily

6.4 Which of the following potentials does not satisfy Laplace's equation?

(a) $V = 2x + 5$
(b) $V = 10\,xy$
(c) $V = r \cos \phi$
(d) $V = \dfrac{10}{r}$
(e) $V = \rho \cos \phi + 10$

6.5 Which of the following is not true?

(a) $-5 \cos 3x$ is a solution to $\phi''(x) + 9\phi(x) = 0$
(b) $10 \sin 2x$ is a solution to $\phi''(x) - 4\phi(x) = 0$
(c) $-4 \cosh 3y$ is a solution to $R''(y) - 9R(y) = 0$
(d) $\sinh 2y$ is a solution to $R''(y) - 4R(y) = 0$
(e) $\dfrac{g''(x)}{g(x)} = -\dfrac{h''(y)}{h(y)} = f(z) = -1$ where $g(x) = \sin x$ and $h(y) = \sinh y$

6.6 If $V_1 = X_1Y_1$ is a product solution of Laplace's equation, which of these are not solutions of Laplace's equation?

(a) $-10X_1Y_1$
(b) $X_1Y_1 + 2xy$
(c) $X_1Y_1 - x + y$
(d) $X_1 + Y_1$
(e) $(X_1 - 2)(Y_1 + 3)$

6.7 The capacitance of a capacitor filled by a linear dielectric is independent of the charge on the plates and the potential difference between the plates.

(a) True
(b) False

6.8 A parallel-plate capacitor connected to a battery stores twice as much charge with a given dielectric as it does with air as dielectric, the susceptibility of the dielectric is

(a) 0
(b) 1
(c) 2
(d) 3
(e) 4

6.9 A potential difference V_o is applied to a mercury column in a cylindrical container. The mercury is now poured into another cylindrical container of half the radius and the same potential difference V_o applied across the ends. As a result of this change of space, the resistance will be increased

(a) 2 times
(b) 4 times
(c) 8 times
(d) 16 times

6.10 Two conducting plates are inclined at an angle 30° to each other with a point charge between them. The number of image charges is

(a) 12
(b) 11
(c) 6
(d) 5
(e) 3

Answers: 6.1a, 6.2c, 6.3a, 6.4c, 6.5b, 6.6d,e, 6.7a, 6.8b, 6.9d, 6.10b.

PROBLEMS

6.1 In free space, $V = 6xy^2z + 8$. At point $P(1, 2, -5)$, find $\mathbf{E}$ and ρ_v.

6.2 Two infinitely large conducting plates are located at $x = 1$ and $x = 4$. The space between them is free space with charge distribution $\frac{x}{6\pi}$ nC/m^3. Find V at $x = 2$ if $V(1) = -50$ V and $V(4) = 50$ V.

6.3 The region between $x = 0$ and $x = d$ is free space and has $\rho_v = \rho_o(x - d)/d$. If $V(x = 0) = 0$ and $V(x = d) = V_o$, find: (a) V and $\mathbf{E}$, (b) the surface charge densities at $x = 0$ and $x = d$.

6.4 Show that the exact solution of the equation

$$\frac{d^2V}{dx^2} = f(x) \qquad 0 < x < L$$

subject to

$$V(x = 0) = V_1 \qquad V(x = L) = V_2$$

is

$$V(x) = \left[V_2 - V_1 - \int_0^L \int_0^\lambda f(\mu)\, d\mu\, d\lambda\right]\frac{x}{L} + V_1 + \int_0^x \int_0^\lambda f(\mu)\, d\mu\, d\lambda$$

6.5 A certain material occupies the space between two conducting slabs located at $y = \pm 2$ cm. When heated, the material emits electrons such that $\rho_v = 50(1 - y^2)$ μC/m^3. If the slabs are both held at 30 kV, find the potential distribution within the slabs. Take $\varepsilon = 3\varepsilon_o$.

6.6 Determine which of the following potential field distributions satisfy Laplace's equation.

(a) $V_1 = x^2 + y^2 - 2z^2 + 10$

(b) $V_2 = \dfrac{1}{(x^2 + y^2 + z^2)^{1/2}}$

(c) $V_3 = \rho z \sin\phi + \rho^2$

(d) $V_4 = \dfrac{10 \sin\theta \sin\phi}{r^2}$

6.7 Show that the following potentials satisfy Laplace's equation.

(a) $V = e^{-5x} \cos 13y \sinh 12z$

(b) $V = \dfrac{z \cos\phi}{\rho}$

(c) $V = \dfrac{30 \cos\theta}{r^2}$

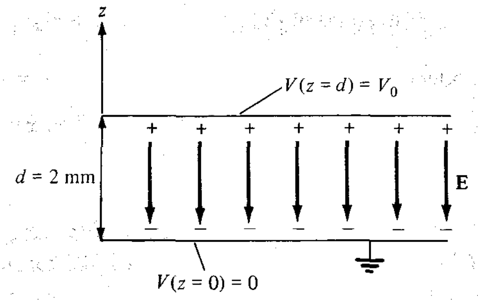


Figure 6.26 For Problem 6.11.

6.8 Show that $\mathbf{E} = (E_x, E_y, E_z)$ satisfies Laplace's equation.

6.9 Let $V = (A \cos nx + B \sin nx)(Ce^{ny} + De^{-ny})$, where A, B, C, and D are constants. Show that V satisfies Laplace's equation.

6.10 The potential field $V = 2x^2yz - y^3z$ exists in a dielectric medium having $\varepsilon = 2\varepsilon_o$. (a) Does V satisfy Laplace's equation? (b) Calculate the total charge within the unit cube $0 < x,y,z < 1$ m.

6.11 Consider the conducting plates shown in Figure 6.26. If $V(z = 0) = 0$ and $V(z = 2 \text{ mm}) = 50$ V, determine V, $\mathbf{E}$, and $\mathbf{D}$ in the dielectric region ($\varepsilon_r = 1.5$) between the plates and ρ_S on the plates.

6.12 The cylindrical capacitor whose cross section is in Figure 6.27 has inner and outer radii of 5 mm and 15 mm, respectively. If $V(\rho = 5 \text{ mm}) = 100$ V and $V(\rho = 15 \text{ mm}) = 0$ V, calculate V, $\mathbf{E}$, and $\mathbf{D}$ at $\rho = 10$ mm and ρ_S on each plate. Take $\varepsilon_r = 2.0$.

6.13 Concentric cylinders $\rho = 2$ cm and $\rho = 6$ cm are maintained at $V = 60$ V and $V = -20$ V, respectively. Calculate V, $\mathbf{E}$, and $\mathbf{D}$ at $\rho = 4$ cm.

6.14 The region between concentric spherical conducting shells $r = 0.5$ m and $r = 1$ m is charge free. If $V(r = 0.5) = -50$ V and $V(r = 1) = 50$ V, determine the potential distribution and the electric field strength in the region between the shells.

6.15 Find V and $\mathbf{E}$ at (3, 0, 4) due to the two conducting cones of infinite extent shown in Figure 6.28.

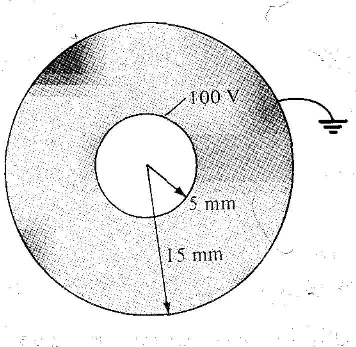


Figure 6.27 Cylindrical capacitor of Problem 6.12.

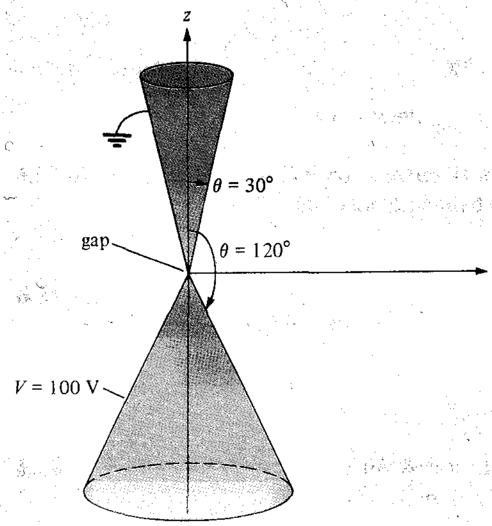


Figure 6.28 Conducting cones of Problem 6.15.

***6.16** The inner and outer electrodes of a diode are coaxial cylinders of radii $a = 0.6$ m and $b = 30$ mm, respectively. The inner electrode is maintained at 70 V while the outer electrode is grounded. (a) Assuming that the length of the electrodes $\ell \gg a, b$ and ignoring the effects of space charge, calculate the potential at $\rho = 15$ mm. (b) If an electron is injected radially through a small hole in the inner electrode with velocity 10^7 m/s, find its velocity at $\rho = 15$ mm.

6.17 Another method of finding the capacitance of a capacitor is using energy considerations, that is

$$C = \frac{2W_E}{V_o^2} = \frac{1}{V_o^2}\int \varepsilon|\mathbf{E}|^2\, dv$$

Using this approach, derive eqs. (6.22), (6.28), and (6.32).

6.18 An electrode with a hyperbolic shape ($xy = 4$) is placed above an earthed right-angle corner as in Figure 6.29. Calculate V and $\mathbf{E}$ at point (1, 2, 0) when the electrode is connected to a 20-V source.

***6.19** Solve Laplace's equation for the two-dimensional electrostatic systems of Figure 6.30 and find the potential $V(x, y)$.

***6.20** Find the potential $V(x, y)$ due to the two-dimensional systems of Figure 6.31.

6.21 By letting $V(\rho, \phi) = R(\rho)\Phi(\phi)$ be the solution of Laplace's equation in a region where $\rho \neq 0$, show that the separated differential equations for R and Φ are

$$R'' + \frac{R'}{\rho} - \frac{\lambda}{\rho^2}R = 0$$

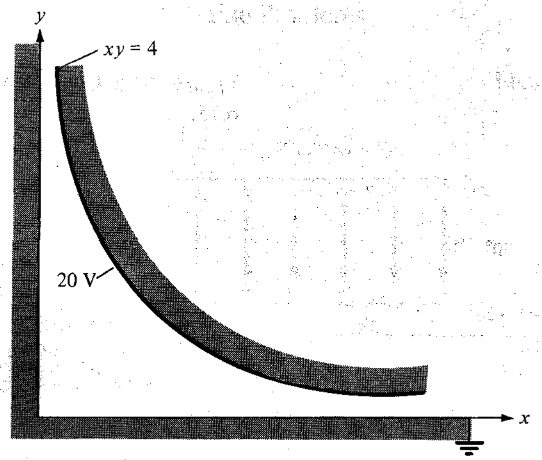


Figure 6.29 For Problem 6.18.

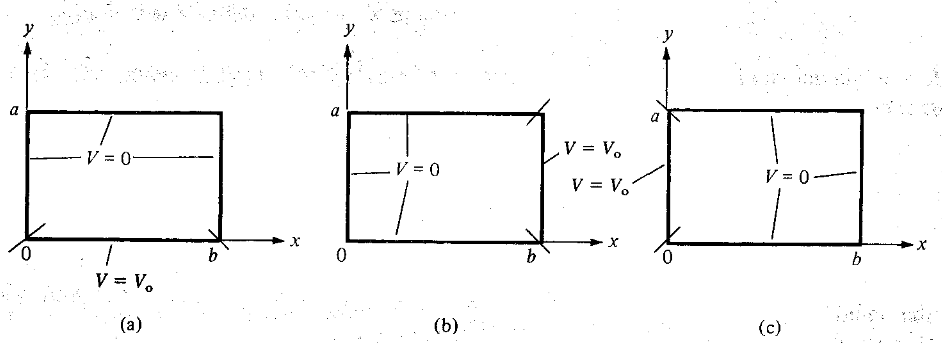


Figure 6.30 For Problem 6.19.

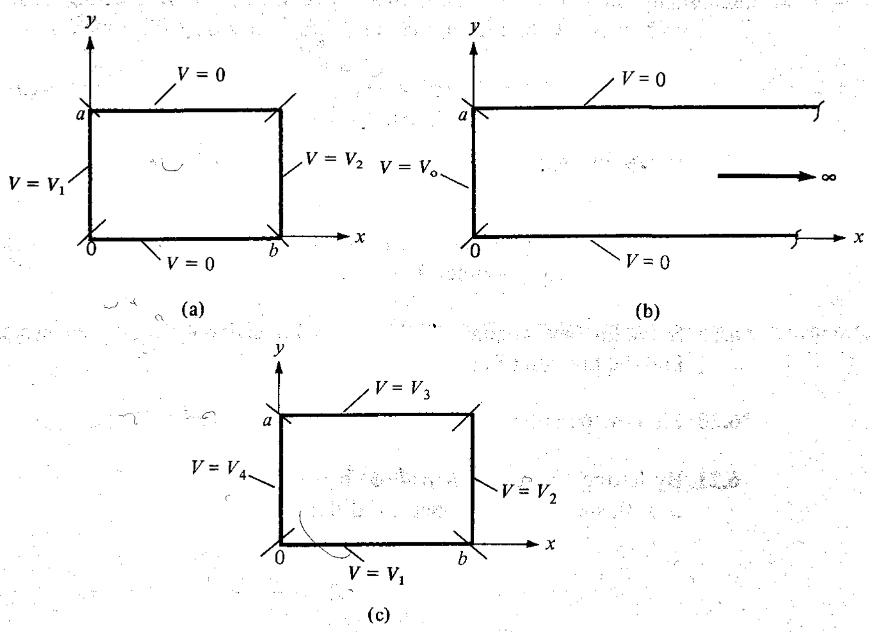


Figure 6.31 For Problem 6.20.

and

$$\Phi'' + \lambda\Phi = 0$$

where λ is the separation constant.

6.22 A potential in spherical coordinates is a function of r and θ but not ϕ. Assuming that $V(r, \theta) = R(r)F(\theta)$, obtain the separated differential equations for R and F in a region for which $\rho_v = 0$.

6.23 Show that the resistance of the bar of Figure 6.17 between the vertical ends located at $\phi = 0$ and $\phi = \pi/2$ is

$$R = \frac{\pi}{2\sigma t \ln b/a}$$

***6.24** Show that the resistance of the sector of a spherical shell of conductivity σ, with cross section shown in Figure 6.32 (where $0 \leq \phi < 2\pi$), between its base is

$$R = \frac{1}{2\pi\sigma(1 - \cos\alpha)}\left[\frac{1}{a} - \frac{1}{b}\right]$$

***6.25** A hollow conducting hemisphere of radius a is buried with its flat face lying flush with the earth surface thereby serving as an earthing electrode. If the conductivity of earth is σ, show that the leakage conductance between the electrode and earth is $2\pi a\sigma$.

6.26 The cross section of an electric fuse is shown in Figure 6.33. If the fuse is made of copper and of thickness 1.5 mm, calculate its resistance.

6.27 In an integrated circuit, a capacitor is formed by growing a silicon dioxide layer ($\varepsilon_r = 4$) of thickness 1 μm over the conducting silicon substrate and covering it with a metal electrode of area S. Determine S if a capacitance of 2 nF is desired.

6.28 The parallel-plate capacitor of Figure 6.34 is quarter-filled with mica ($\varepsilon_r = 6$). Find the capacitance of the capacitor.

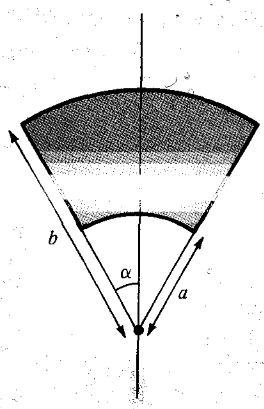


Figure 6.32 For Problem 6.24.

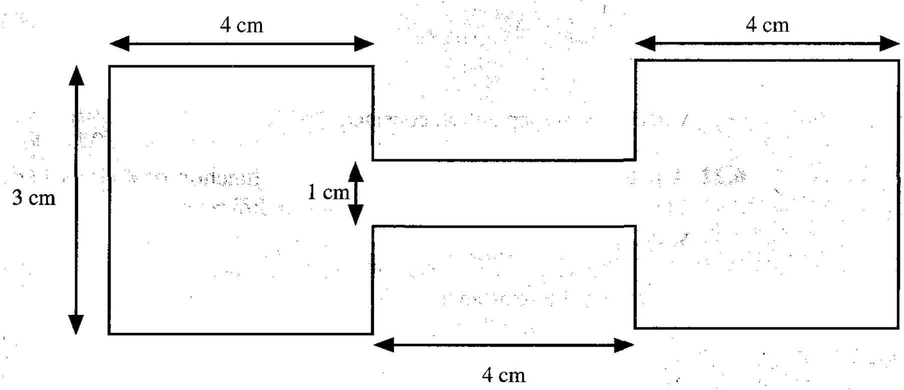


Figure 6.33 For Problem 6.26.

***6.29** An air-filled parallel plate capacitor of length L, width a, and plate separation d has its plates maintained at constant potential difference V_o. If a dielectric slab of dielectric constant ε_r is slid between the plates and is withdrawn until only a length x remains between the plates as in Figure 6.35, show that the force tending to restore the slab to its original position is

$$F = \frac{\varepsilon_o(\varepsilon_r - 1)\, a\, V_o^2}{2d}$$

6.30 A parallel-plate capacitor has plate area 200 cm^2 and plate separation 3 mm. The charge density is 1 μC/m^2 with air as dielectric. Find

(a) The capacitance of the capacitor
(b) The voltage between the plates
(c) The force with which the plates attract each other

6.31 Two conducting plates are placed at $z = -2$ cm and $z = 2$ cm and are, respectively, maintained at potentials 0 and 200 V. Assuming that the plates are separated by a polypropylene ($\varepsilon = 2.25\varepsilon_o$). Calculate: (a) the potential at the middle of the plates, (b) the surface charge densities at the plates.

6.32 Two conducting parallel plates are separated by a dielectric material with $\varepsilon = 5.6\varepsilon_o$ and thickness 0.64 mm. Assume that each plate has an area of 80 cm^2. If the potential field distribution between the plates is $V = 3x + 4y - 12z + 6$ kV, determine: (a) the capacitance of the capacitor, (b) the potential difference between the plates.

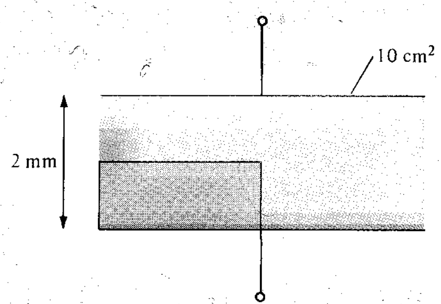


Figure 6.34 For Problem 6.28.

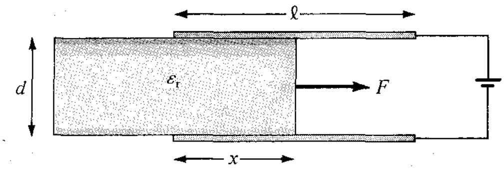


Figure 6.35 For Problem 6.29.

6.33 The space between spherical conducting shells $r = 5$ cm and $r = 10$ cm is filled with a dielectric material for which $\varepsilon = 2.25\varepsilon_o$. The two shells are maintained at a potential difference of 80 V. (a) Find the capacitance of the system. (b) Calculate the charge density on shell $r = 5$ cm.

6.34 Concentric shells $r = 20$ cm and $r = 30$ cm are held at $V = 0$ and $V = 50$, respectively. If the space between them is filled with dielectric material ($\varepsilon = 3.1\varepsilon_o$, $\sigma = 10^{-12}$ S/m), find: (a) V, **E**, and **D**, (b) the charge densities on the shells, (c) the leakage resistance.

6.35 A spherical capacitor has inner radius a and outer radius d. Concentric with the spherical conductors and lying between them is a spherical shell of outer radius c and inner radius b. If the regions $d < r < c$, $c < r < b$, and $b < r < a$ are filled with materials with permittivites ε_1, ε_2, and ε_3, respectively, determine the capacitance of the system.

6.36 Determine the capacitance of a conducting sphere of radius 5 cm deeply immersed in sea water ($\varepsilon_r = 80$).

6.37 A conducting sphere of radius 2 cm is surrounded by a concentric conducting sphere of radius 5 cm. If the space between the spheres is filled with sodium chloride ($\varepsilon_r = 5.9$), calculate the capacitance of the system.

***6.38** In an ink-jet printer the drops are charged by surrounding the jet of radius 20 μm with a concentric cylinder of radius 600 μm as in Figure 6.36. Calculate the minimum voltage required to generate a charge 50 fC on the drop if the length of the jet inside the cylinder is 100 μm. Take $\varepsilon = \varepsilon_o$.

6.39 A given length of a cable, the capacitance of which is 10 μF/km with a resistance of insulation of 100 MΩ/km, is charged to a voltage of 100 V. How long does it take the voltage to drop to 50 V?

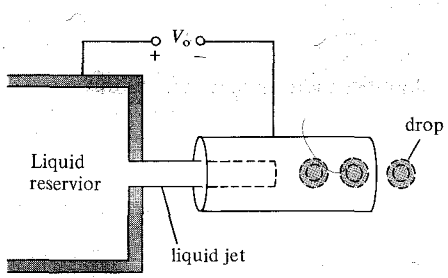


Figure 6.36 Simplified geometry of an ink-jet printer; for Problem 6.38.

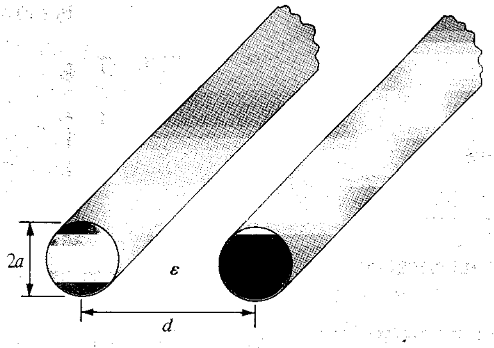


Figure 6.37 For Problem 6.40.

6.40 The capacitance per unit length of a two-wire transmission line shown in Figure 6.37 is given by

$$C = \frac{\pi\varepsilon}{\cosh^{-1}\left[\dfrac{d}{2a}\right]}$$

Determine the conductance per unit length.

***6.41** A spherical capacitor has an inner conductor of radius a carrying charge Q and maintained at zero potential. If the outer conductor contracts from a radius b to c under internal forces, prove that the work performed by the electric field as a result of the contraction is

$$W = \frac{Q^2(b - c)}{8\pi\varepsilon bc}$$

***6.42** A parallel-plate capacitor has its plates at $x = 0, d$ and the space between the plates is filled with an inhomogeneous material with permittivity $\varepsilon = \varepsilon_o\left(1 + \dfrac{x}{d}\right)$. If the plate at $x = d$ is maintained at V_o while the plate at $x = 0$ is grounded, find:

(a) V and $\mathbf{E}$

(b) $\mathbf{P}$

(c) ρ_{ps} at $x = 0, d$

6.43 A spherical capacitor has inner radius a and outer radius b and filled with an inhomogeneous dielectric with $\varepsilon = \varepsilon_o k/r^2$. Show that the capacitance of the capacitor is

$$C = \frac{4\pi\varepsilon_o k}{b - a}$$

6.44 A cylindrical capacitor with inner radius a and outer radius b is filled with an inhomogeneous dielectric having $\varepsilon = \varepsilon_o k/\rho$, where k is a constant. Calculate the capacitance per unit length of the capacitor.

6.45 If the earth is regarded a spherical capacitor, what is its capacitance? Assume the radius of the earth to be approximately 6370 km.

6.46 A point charge of 10 nC is located at point $P(0, 0, 3)$ while the conducting plane $z = 0$ is grounded. Calculate

(a) V and $\mathbf{E}$ at $R(6, 3, 5)$

(b) The force on the charge due to induced charge on the plane.

6.47 Two point charges of 3 nC and -4 nC are placed, respectively, at (0, 0, 1 m) and (0, 0, 2 m) while an infinite conducting plane is at $z = 0$. Determine

(a) The total charge induced on the plane

(b) The magnitude of the force of attraction between the charges and the plane

6.48 Two point charges of 50 nC and -20 nC are located at $(-3, 2, 4)$ and $(1, 0, 5)$ above the conducting ground plane $z = 2$. Calculate (a) the surface charge density at $(7, -2, 2)$, (b) $\mathbf{D}$ at $(3, 4, 8)$, and (c) $\mathbf{D}$ at $(1, 1, 1)$.

***6.49** A point charge of 10 μC is located at (1, 1, 1), and the positive portions of the coordinate planes are occupied by three mutually perpendicular plane conductors maintained at zero potential. Find the force on the charge due to the conductors.

6.50 A point charge Q is placed between two earthed intersecting conducting planes that are inclined at 45° to each other. Determine the number of image charges and their locations.

6.51 Infinite line $x = 3, z = 4$ carries 16 nC/m and is located in free space above the conducting plane $z = 0$. (a) Find $\mathbf{E}$ at $(2, -2, 3)$. (b) Calculate the induced surface charge density on the conducting plane at $(5, -6, 0)$.

6.52 In free space, infinite planes $y = 4$ and $y = 8$ carry charges 20 nC/m^2 and 30 nC/m^2, respectively. If plane $y = 2$ is grounded, calculate $\mathbf{E}$ at $P(0, 0, 0)$ and $Q(-4, 6, 2)$.