5

FOURIER SERIES

Our first goal in this key chapter is to find the coefficients in a Fourier series. In Section 5.3 we introduce the idea of orthogonality of functions and we show how the different varieties of Fourier series can be treated in a unified fashion. In Section 5.4 we state the basic completeness (convergence) theorems. Proofs are given in Section 5.5. The final section is devoted to the treatment of inhomogeneous boundary conditions. Joseph Fourier developed his ideas on the convergence of trigonometric series while studying heat flow. His 1807 paper was rejected by other scientists as too imprecise and was not published until 1822.

5.1 THE COEFFICIENTS

In Chapter 4 we have found Fourier series of several types. How do we find the coefficients? Luckily, there is a very beautiful, conceptual formula for them.

Let us begin with the *Fourier sine series*

$$\boxed{\phi(x) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}} \tag{1}$$

in the interval $(0, l)$. [It turns out that this infinite series converges to $\phi(x)$ for $0 < x < l$, but let's postpone further discussion of the delicate question of convergence for the time being.] The first problem we tackle is to try to find the coefficients A_n if $\phi(x)$ is a given function. The key observation is that the sine functions have the wonderful property that

$$\boxed{\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx = 0 \quad \text{if } m \neq n,} \tag{2}$$

m and n being positive integers. This can be verified directly by integration. [Historically, (1) was first discovered by a horrible expansion in Taylor series!]

Proof of (2). We use the trig identity

$$\sin a \sin b = \tfrac{1}{2}\cos(a-b) - \tfrac{1}{2}\cos(a+b).$$

Therefore, the integral equals

$$\frac{l}{2(m-n)\pi} \sin \frac{(m-n)\pi x}{l}\bigg|_0^l - [\text{same with } (m+n)]$$

if $m \neq n$. This is a linear combination of $\sin(m \pm n)\pi$ and $\sin 0$, and so it vanishes. □

The far-reaching implications of this observation are astounding. Let's *fix* m, multiply (1) by $\sin(m\pi x/l)$, and integrate the series (1) term by term to get

$$\begin{aligned}\int_0^l \phi(x) \sin \frac{m\pi x}{l}\, dx &= \int_0^l \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx \\ &= \sum_{n=1}^{\infty} A_n \int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l}\, dx.\end{aligned}$$

All but one term in this sum vanishes, namely the one with $n = m$ (n just being a "dummy" index that takes on all integer values ≥ 1). Therefore, we are left with the single term

$$A_m \int_0^l \sin^2 \frac{m\pi x}{l}\, dx, \tag{3}$$

which equals $\frac{1}{2} l A_m$ by explicit integration. Therefore,

$$\boxed{A_m = \frac{2}{l} \int_0^l \phi(x) \sin \frac{m\pi x}{l}\, dx.} \tag{4}$$

This is the famous *formula for the Fourier coefficients* in the series (1). That is, *if* $\phi(x)$ has an expansion (1), *then* the coefficients must be given by (4).

These are the only possible coefficients in (1). However, the basic question still remains whether (1) is in fact valid with these values of the coefficients. This is a question of convergence, and we postpone it until Section 5.4.

APPLICATION TO DIFFUSIONS AND WAVES

Going back to the diffusion equation with Dirichlet boundary conditions, the formula (4) provides the final ingredient in the solution formula for arbitrary initial data $\phi(x)$.

As for the wave equation with Dirichlet conditions, the initial data consist of a pair of functions $\phi(x)$ and $\psi(x)$ with expansions (4.1.10) and (4.1.11). The coefficients A_m in (4.1.9) are given by (4), while for the same reason the coefficients B_m are given by the similar formula

$$\frac{m\pi c}{l} B_m = \frac{2}{l} \int_0^l \psi(x) \sin \frac{m\pi x}{l}\, dx. \tag{5}$$

FOURIER COSINE SERIES

Next let's take the case of the cosine series, which corresponds to the Neumann boundary conditions on $(0, l)$. We write it as

$$\boxed{\phi(x) = \frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.} \tag{6}$$

Again we can verify the magical fact that

$$\int_0^l \cos \frac{n\pi x}{l} \cos \frac{m\pi x}{l}\, dx = 0 \qquad \text{if } m \neq n$$

where m and n are nonnegative integers. (Verify it!) By exactly the same method as above, but with sines replaced by cosines, we get

$$\int_0^l \phi(x) \cos \frac{m\pi x}{l}\, dx = A_m \int_0^l \cos^2 \frac{m\pi x}{l}\, dx = \frac{1}{2} l A_m$$

if $m \neq 0$. For the case $m = 0$, we have

$$\int_0^l \phi(x) \cdot 1\, dx = \frac{1}{2} A_0 \int_0^l 1^2\, dx = \frac{1}{2} l A_0.$$

Therefore, for all *nonnegative* integers m, we have the formula for the coefficients of the cosine series

$$\boxed{A_m = \frac{2}{l} \int_0^l \phi(x) \cos \frac{m\pi x}{l}\, dx.} \tag{7}$$

[This is the reason for putting the $\frac{1}{2}$ in front of the constant term in (6).]

FULL FOURIER SERIES

The full Fourier series, or simply the Fourier series, of $\phi(x)$ on the interval $-l < x < l$, is defined as

$$\boxed{\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right).} \tag{8}$$

Watch out: The interval is twice as long! The eigenfunctions now are all the functions $\{1, \cos(n\pi x/l), \sin(n\pi x/l)\}$, where $n = 1, 2, 3, \ldots$. Again we have the same wonderful coincidence: Multiply any two different eigenfunctions and integrate over the interval and you get zero! That is,

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = 0 \qquad \text{for all } n, m$$

$$\int_{-l}^{l} \cos\frac{n\pi x}{l}\cos\frac{m\pi x}{l}\,dx = 0 \qquad \text{for } n \neq m$$

$$\int_{-l}^{l} \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\,dx = 0 \qquad \text{for } n \neq m$$

$$\int_{-l}^{l} 1\cdot\cos\frac{n\pi x}{l}\,dx = 0 = \int_{-l}^{l} 1\cdot\sin\frac{m\pi x}{l}\,dx.$$

Therefore, the same procedure will work to find the coefficients. We also calculate the integrals of the squares

$$\int_{-l}^{l} \cos^2\frac{n\pi x}{l}\,dx = l = \int_{-l}^{l} \sin^2\frac{n\pi x}{l}\,dx \quad \text{and} \quad \int_{-l}^{l} 1^2\,dx = 2l.$$

(Verify these integrals too!) Then we end up with the formulas

$$A_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\cos\frac{n\pi x}{l}\,dx \quad (n = 0, 1, 2, \ldots) \tag{9}$$

$$B_n = \frac{1}{l}\int_{-l}^{l} \phi(x)\sin\frac{n\pi x}{l}\,dx \quad (n = 1, 2, 3, \ldots) \tag{10}$$

for the coefficients of the full Fourier series. Note that these formulas are *not* exactly the same as (4) and (7).

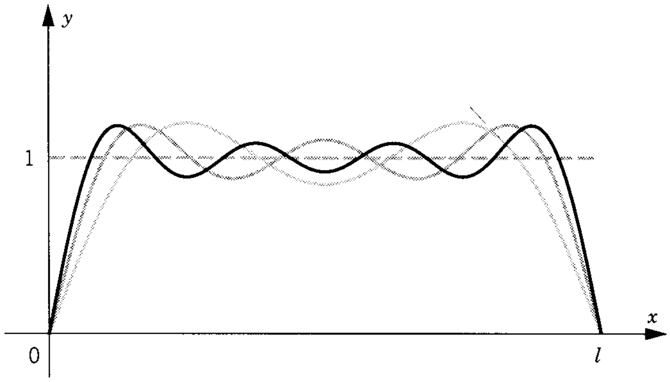

Figure 1

Example 1.

Let $\phi(x) \equiv 1$ in the interval $[0, l]$. It has a Fourier sine series with coefficients

$$A_m = \frac{2}{l}\int_0^l \sin\frac{m\pi x}{l}\,dx = -\frac{2}{m\pi}\cos\frac{m\pi x}{l}\bigg|_0^l$$
$$= \frac{2}{m\pi}(1-\cos m\pi) = \frac{2}{m\pi}[1-(-1)^m].$$

Thus $A_m = 4/m\pi$ if m is odd, and $A_m = 0$ if m is even. Thus

$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right) \tag{11}$$

in $(0, l)$. (The factor $4/\pi$ is pulled out just for notational convenience.) See Figure 1 for a sketch of the first few partial sums. □

Example 2.

The same function $\phi(x) \equiv 1$ has a Fourier cosine series with coefficients

$$A_m = \frac{2}{l}\int_0^l \cos\frac{m\pi x}{l}\,dx = \frac{2}{m\pi}\sin\frac{m\pi x}{l}\bigg|_0^l$$
$$= \frac{2}{m\pi}(\sin m\pi - \sin 0) = 0 \qquad \text{for } m \neq 0.$$

So there is only one nonzero coefficient, namely, the one for $m = 0$. The Fourier cosine series is therefore trivial:

$$1 = 1 + 0\cos\frac{\pi x}{l} + 0\cos\frac{2\pi x}{l} + \cdots.$$

This is perfectly natural since the sum $1 = 1 + 0 + 0 + 0 + \cdots$ is obvious and the Fourier cosine expansion is unique. □

Example 3.

Let $\phi(x) \equiv x$ in the interval $(0, l)$. Its Fourier sine series has the coefficients

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l x \sin\frac{m\pi x}{l}\,dx \\ &= -\frac{2x}{m\pi}\cos\frac{m\pi x}{l} + \frac{2l}{m^2\pi^2}\sin\frac{m\pi x}{l}\bigg|_0^l \\ &= -\frac{2l}{m\pi}\cos m\pi + \frac{2l}{m^2\pi^2}\sin m\pi = (-1)^{m+1}\frac{2l}{m\pi}. \end{aligned}$$

Thus in $(0, l)$ we have

$$x = \frac{2l}{\pi}\left(\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} - \cdots\right). \tag{12}$$

□

Example 4.

Let $\phi(x) \equiv x$ in the interval $[0, l]$. Its Fourier cosine series has the coefficients

$$\begin{aligned} A_0 &= \frac{2}{l}\int_0^l x\,dx = l \\ A_m &= \frac{2}{l}\int_0^l x\cos\frac{m\pi x}{l}\,dx \\ &= \frac{2x}{m\pi}\sin\frac{m\pi x}{l} + \frac{2l}{m^2\pi^2}\cos\frac{m\pi x}{l}\bigg|_0^l \\ &= \frac{2l}{m\pi}\sin m\pi + \frac{2l}{m^2\pi^2}(\cos m\pi - 1) = \frac{2l}{m^2\pi^2}[(-1)^m - 1] \\ &= \frac{-4l}{m^2\pi^2} \quad \text{for } m \text{ odd,} \quad \text{and} \quad 0 \quad \text{for } m \text{ even.} \end{aligned}$$

Thus in $(0, l)$ we have

$$x = \frac{l}{2} - \frac{4l}{\pi^2}\left(\cos\frac{\pi x}{l} + \frac{1}{9}\cos\frac{3\pi x}{l} + \frac{1}{25}\cos\frac{5\pi x}{l} + \cdots\right). \tag{13}$$

□

Example 5.

Let $\phi(x) \equiv x$ in the interval $[-l, l]$. Its full Fourier series has the coefficients

$$\begin{aligned}
A_0 &= \frac{1}{l}\int_{-l}^{l} x\,dx = 0 \\
A_m &= \frac{1}{l}\int_{-l}^{l} x\cos\frac{m\pi x}{l}\,dx \\
&= \frac{x}{m\pi}\sin\frac{m\pi x}{l} + \frac{l}{m^2\pi^2}\cos\frac{m\pi x}{l}\bigg|_{-l}^{l} \\
&= \frac{l}{m^2\pi^2}(\cos m\pi - \cos(-m\pi)) = 0 \\
B_m &= \frac{1}{l}\int_{-l}^{l} x\sin\frac{m\pi x}{l}\,dx \\
&= \frac{-x}{m\pi}\cos\frac{m\pi x}{l} + \frac{l}{m^2\pi^2}\sin\frac{m\pi x}{l}\bigg|_{-l}^{l} \\
&= \frac{-l}{m\pi}\cos m\pi + \frac{-l}{m\pi}\cos(-m\pi) = (-1)^{m+1}\frac{2l}{m\pi}.
\end{aligned}$$

This gives us exactly the same series as (12), except that it is supposed to be valid in $(-l, l)$, which is not a surprising result because both sides of (12) are odd. □

Example 6.

Solve the problem

$$\begin{gathered}
u_{tt} = c^2 u_{xx} \\
u(0, t) = u(l, t) = 0 \\
u(x, 0) = x, \quad u_t(x, 0) = 0.
\end{gathered}$$

From Section 4.1 we know that $u(x, t)$ has an expansion

$$u(x, t) = \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

Differentiating with respect to time yields

$$u_t(x, t) = \sum_{n=1}^{\infty}\frac{n\pi c}{l}\left(-A_n\sin\frac{n\pi ct}{l} + B_n\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}.$$

Setting $t = 0$, we have

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}$$

so that all the $B_n = 0$. Setting $t = 0$ in the expansion of $u(x, t)$, we have

$$x = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}.$$

This is exactly the series of Example 3. Therefore, the complete solution is

$$u(x, t) = \frac{2l}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}. \qquad \square$$

EXERCISES

1. In the expansion $1 = \sum_{n \text{ odd}} (4/n\pi) \sin n\pi$, valid for $0 < x < \pi$, put $x = \pi/4$ to calculate the sum

$$\left(1 - \tfrac{1}{5} + \tfrac{1}{9} - \tfrac{1}{13} + \cdots\right) + \left(\tfrac{1}{3} - \tfrac{1}{7} + \tfrac{1}{11} - \tfrac{1}{15} + \cdots\right)$$
$$= 1 + \tfrac{1}{3} - \tfrac{1}{5} - \tfrac{1}{7} + \tfrac{1}{9} + \cdots$$

(*Hint:* Since each of the series converges, they can be combined as indicated. However, they cannot be arbitrarily rearranged because they are only conditionally, not absolutely, convergent.)

2. Let $\phi(x) \equiv x^2$ for $0 \le x \le 1 = l$.
   (a) Calculate its Fourier sine series.
   (b) Calculate its Fourier cosine series.
3. Consider the function $\phi(x) \equiv x$ on $(0, l)$. On the same graph, *sketch* the following functions.
   (a) The sum of the first three (nonzero) terms of its Fourier sine series.
   (b) The sum of the first three (nonzero) terms of its Fourier cosine series.
4. Find the Fourier cosine series of the function $|\sin x|$ in the interval $(-\pi, \pi)$. Use it to find the sums

$$\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{4n^2 - 1}.$$

5. Given the Fourier sine series of $\phi(x) \equiv x$ on $(0, l)$. Assume that the series can be integrated term by term, a fact that will be shown later.
   (a) Find the Fourier cosine series of the function $x^2/2$. Find the constant of integration that will be the first term in the cosine series.

(b) Then by setting $x = 0$ in your result, find the *sum* of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}.$$

6. (a) By the same method, find the sine series of x^3.
   (b) Find the cosine series of x^4.
7. Put $x = 0$ in Exercise 6(b) to deduce the sum of the series

$$\sum_{1}^{\infty} \frac{(-1)^n}{n^4}.$$

8. A rod has length $l = 1$ and constant $k = 1$. Its temperature satisfies the heat equation. Its left end is held at temperature 0, its right end at temperature 1. Initially (at $t = 0$) the temperature is given by

$$\phi(x) = \begin{cases} \dfrac{5x}{2} & \text{for } 0 < x < \frac{2}{3} \\ 3 - 2x & \text{for } \frac{2}{3} < x < 1. \end{cases}$$

   Find the solution, including the coefficients. (*Hint:* First find the equilibrium solution $U(x)$, and then solve the heat equation with initial condition $u(x, 0) = \phi(x) - U(x)$.)
9. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \pi$, with the boundary conditions $u_x(0, t) = u_x(\pi, t) = 0$ and the initial conditions $u(x, 0) = 0,\ u_t(x, 0) = \cos^2 x$. (*Hint:* See (4.2.7).)
10. A string (of tension T and density ρ) with fixed ends at $x = 0$ and $x = l$ is hit by a hammer so that $u(x, 0) = 0$, and $\partial u/\partial t(x, 0) = V$ in $[-\delta + \frac{1}{2}l, \delta + \frac{1}{2}l]$ and $\partial u/\partial t(x, 0) = 0$ elsewhere. Find the solution explicitly in series form. Find the energy

$$E_n(h) = \frac{1}{2} \int_0^l \left[\rho \left(\frac{\partial h}{\partial t} \right)^2 + T \left(\frac{\partial h}{\partial x} \right)^2 \right] dx$$

   of the nth harmonic $h = h_n$. Conclude that if δ is small (a concentrated blow), each of the first few overtones has almost as much energy as the fundamental. We could say that the tone is saturated with overtones.
11. On a string with fixed ends, show that if the center of a hammer blow is exactly at a node of the nth harmonic (a place where the nth eigenfunction vanishes), the nth overtone is absent from the solution.

5.2 EVEN, ODD, PERIODIC, AND COMPLEX FUNCTIONS

Each of the three kinds of Fourier series (sine, cosine, and full) of any given function $\phi(x)$ is now determined by the formula for its coefficients given in Section 5.1. We shall see shortly that almost any function $\phi(x)$ defined on the interval $(0, l)$ is the sum of its Fourier sine series and is *also* the sum of its Fourier cosine series. Almost any function defined on the interval $(-l, l)$ is the sum of its full Fourier series. Each of these series converges inside the interval, but not necessarily at the endpoints.

The concepts of oddness, evenness, and periodicity are closely related to the three kinds of Fourier series.

A function $\phi(x)$ that is defined for $-\infty < x < \infty$ is called *periodic* if there is a number $p > 0$ such that

$$\phi(x + p) = \phi(x) \qquad \text{for all } x. \tag{1}$$

A number p for which this is true is called a *period* of $\phi(x)$. The graph of the function repeats forever horizontally. For instance, $\cos x$ has period 2π, $\cos \lambda x$ has period $2\pi/\lambda$, and $\tan x$ has period π. Note that if $\phi(x)$ has period p, then $\phi(x + np) = \phi(x)$ for all x and for all integers n. (Why?) The sum of two functions of period p has period p. Notice that if $\phi(x)$ has period p, then $\int_a^{a+p} \phi(x)\,dx$ does not depend on a. (Why?)

For instance, the function $\cos(mx) + \sin 2mx$ is the sum of functions of periods $2\pi/m$ and π/m and therefore itself has period $2\pi/m$, the larger of the two.

If a function is defined only on an interval of length p, it can be extended in only one way to a function of period p. The situation we care about for Fourier series is that of a function defined on the interval $-l < x < l$. Its *periodic extension* is

$$\phi_{\text{per}}(x) = \phi(x - 2lm) \qquad \text{for} \quad -l + 2lm < x < +l + 2lm \tag{2}$$

for all integers m. This definition does not specify what the periodic extension is at the endpoints $x = l + 2lm$. In fact, the extension has jumps at these points unless the one-sided limits are equal: $\phi(l-) = \phi(-l+)$ (see Figure 1). (See Section A.1 for the definition of one-sided limits.)

Since each term in the full Fourier series (5.1.8) has period $2l$, its sum (if it converges) also has to have period $2l$. Therefore, the full Fourier series can be regarded *either* as an expansion of an arbitrary function on the interval

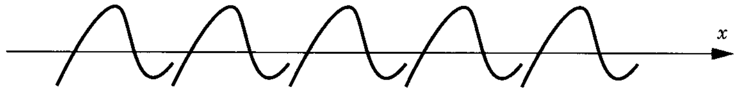

Figure 1

$(-l, l)$ *or* as an expansion of a periodic function of *period 2l* defined on the whole line $-\infty < x < +\infty$. □

An *even function* is a function that satisfies the equation

$$\phi(-x) = \phi(x). \tag{3}$$

That just means that its graph $y = \phi(x)$ is symmetric with respect to the y axis. Thus the left and right halves of the graph are mirror images of each other. To make sense out of (3), we require that $\phi(x)$ be defined on some interval $(-l, +l)$ which is symmetric around $x = 0$.

An *odd function* is a function that satisfies the equation

$$\phi(-x) = -\phi(x). \tag{4}$$

That just means that its graph $y = \phi(x)$ is symmetric with respect to the origin. To make sense out of (4), we again require that $\phi(x)$ be defined on some interval $(-l, +l)$ which is symmetric around $x = 0$.

A monomial x^n is an even function if n is even and is an odd function if n is odd. The functions $\cos x$, $\cosh x$, and any function of x^2 are even functions. The functions $\sin x$, $\tan x$, and $\sinh x$ are odd functions. In fact, the products of functions follow the usual rules: even $\times$ even $=$ even, odd $\times$ odd $=$ even, odd $\times$ even $=$ odd. The sum of two odd functions is again odd, and the sum of two evens is even.

But the sum of an even and an odd function can be anything. Proof: Let $f(x)$ be any function at all defined on $(-l, l)$. Let $\phi(x) = \frac{1}{2}[f(x) + f(-x)]$ and $\psi(x) = \frac{1}{2}[f(x) - f(-x)]$. Then we easily check that $f(x) = \phi(x) + \psi(x)$, that $\phi(x)$ is even and that $\psi(x)$ is odd. The functions ϕ and ψ are called the even and odd parts of f, respectively. For instance, cosh and sinh are the even and odd parts of exp since: $e^x = \cosh x + \sinh x$. If $p(x)$ is any polynomial, its even part is the sum of its even terms, and its odd part is the sum of its odd terms.

Integration and differentiation change the parity (evenness or oddness) of a function. That is, if $\phi(x)$ is even, then both $d\phi/dx$ and $\int_0^x \phi(s)\,ds$ are odd. If $\phi(x)$ is odd, then the derivative and integral are even. (Note that the lower limit of integration is at the origin.)

The graph of an odd function $\phi(x)$ must pass through the origin since $\phi(0) = 0$ follows directly from (4) by putting $x = 0$. The graph of an even function $\phi(x)$ must cross the y axis horizontally, $\phi'(x) = 0$, since the derivative is odd (provided the derivative exists).

Example 1.

$\tan x$ is the product of an odd function ($\sin x$) and an even function ($1/\cos x$), both of period 2π. Therefore $\tan x$ is an odd and periodic function. But notice that its smallest period is π, not 2π. Its derivative $\sec^2 x$ is necessarily even and periodic; it has period π. The dilated function $\tan ax$ is also odd and periodic and has period π/a for any $a > 0$. □

Definite integrals around symmetric intervals have the useful properties:

$$\int_{-l}^{l} (\text{odd})\, dx = 0 \qquad \text{and} \qquad \int_{-l}^{l} (\text{even})\, dx = 2\int_{0}^{l} (\text{even})\, dx. \tag{5}$$

Given any function defined on the interval (0, l), it can be extended in only one way to be even or odd. The *even extension* of $\phi(x)$ is defined as

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{for} \quad 0 < x < l \\ \phi(-x) & \text{for} \quad -l < x < 0. \end{cases} \tag{6}$$

This is just the mirror image. The even extension is not necessarily defined at the origin.

Its *odd extension* is

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for} \quad 0 < x < l \\ -\phi(-x) & \text{for} \quad -l < x < 0 \\ 0 & \text{for} \quad x = 0. \end{cases} \tag{7}$$

This is its image through the origin.

FOURIER SERIES AND BOUNDARY CONDITIONS

Now let's return to the Fourier sine series. Each of its terms, $\sin(n\pi x/l)$, is an odd function. Therefore, its sum (if it converges) also has to be odd. Furthermore, each of its terms has period 2l, so that the same has to be true of its sum. Therefore, *the Fourier sine series can be regarded as an expansion of an arbitrary function that is odd and has period 2l defined on the whole line* $-\infty < x < +\infty$.

Similarly, since all the cosine functions are even, *the Fourier cosine series can be regarded as an expansion of an arbitrary function which is even and has period 2l defined on the whole line* $-\infty < x < \infty$.

From what we saw in Section 5.1, these concepts therefore have the following relationship to boundary conditions:

$u(0, t) = u(l, t) = 0$: Dirichlet BCs correspond to the odd extension. (8)

$u_x(0, t) = u_x(l, t) = 0$: Neumann BCs correspond to the even extension. (9)

$u(l, t) = u(-l, t)$, $u_x(l, t) = u_x(-l, t)$: Periodic BCs correspond to the periodic extension. (10)

THE COMPLEX FORM OF THE FULL FOURIER SERIES

The eigenfunctions of $-d^2/dx^2$ on $(-l, l)$ with the periodic boundary conditions are $\sin(n\pi x/l)$ and $\cos(n\pi x/l)$. But recall the *DeMoivre formulas*, which express the sine and cosine in terms of the complex exponentials:

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \text{and} \qquad \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}. \tag{11}$$

Therefore, instead of sine and cosine, we could use $e^{+in\pi x/l}$ and $e^{-in\pi x/l}$ as an alternative pair. But watch out: They're complex! If we do that, the collection of trigonometric functions $\{\sin n\theta, \cos n\theta\}$ is replaced by the collection of complex exponentials

$$\{1, e^{+i\pi x/l}, e^{+i2\pi x/l}, \ldots, e^{-i\pi x/l}, e^{-i2\pi x/l}, \ldots\}.$$

In other words, we get $\{e^{in\pi x/l}\}$, where n is *any positive or negative* integer.

We should therefore be able to write the full Fourier series in the complex form

$$\boxed{\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/l}.} \tag{12}$$

This is the sum of two infinite series, one going from $n = 0$ to $+\infty$ and one going from $n = -1$ to $-\infty$. The magical fact in this case is

$$\begin{aligned}\int_{-l}^{l} e^{in\pi x/l} e^{-im\pi x/l} dx &= \int_{-l}^{l} e^{i(n-m)\pi x/l} dx \\ &= \frac{l}{i\pi\,(n-m)}[e^{i(n-m)\pi} - e^{i(m-n)\pi}] \\ &= \frac{l}{i\pi\,(n-m)}[(-1)^{n-m} - (-1)^{m-n}] = 0\end{aligned}$$

provided that $n \neq m$. Notice the extra minus sign in the second exponent of the first integral. When $n = m$, we have

$$\int_{-l}^{l} e^{i(n-n)\pi x/l} dx = \int_{-l}^{l} 1\, dx = 2l.$$

It follows by the method of Section 5.1 that the coefficients are given by the formula

$$\boxed{c_n = \frac{1}{2l}\int_{-l}^{l} \phi(x)\, e^{-in\pi x/l} dx.} \tag{13}$$

The complex form is sometimes more convenient in calculations than the real form with sines and cosines. But it really is just the same series written in a different form.

EXERCISES

1. For each of the following functions, state whether it is even or odd or periodic. If periodic, what is its smallest period?
   (a) $\sin ax \quad (a > 0)$
   (b) $e^{ax} \quad (a > 0)$
   (c) $x^m \quad (m = \text{integer})$
   (d) $\tan x^2$
   (e) $|\sin(x/b)| \quad (b > 0)$
   (f) $x \cos ax \quad (a > 0)$
2. Show that $\cos x + \cos \alpha x$ is periodic if α is a rational number. What is its period?
3. Prove property (5) concerning the integrals of even and odd functions.
4. (a) Use (5) to prove that if $\phi(x)$ is an odd function, its full Fourier series on $(-l, l)$ has only sine terms.
   (b) Also, if $\phi(x)$ is an even function, its full Fourier series on $(-l, l)$ has only cosine terms. (*Hint:* Don't use the series directly. Use the formulas for the coefficients to show that every second coefficient vanishes.)
5. Show that the Fourier sine series on $(0, l)$ can be derived from the full Fourier series on $(-l, l)$ as follows. Let $\phi(x)$ be any (continuous) function on $(0, l)$. Let $\tilde{\phi}(x)$ be its odd extension. Write the full series for $\tilde{\phi}(x)$ on $(-l, l)$. [Assume that its sum is $\tilde{\phi}(x)$.] By Exercise 4, this series has only sine terms. Simply restrict your attention to $0 < x < l$ to get the sine series for $\phi(x)$.
6. Show that the cosine series on $(0, l)$ can be derived from the full series on $(-l, l)$ by using the even extension of a function.
7. Show how the full Fourier series on $(-l, l)$ can be derived from the full series on $(-\pi, \pi)$ by changing variables $w = (\pi/l)x$. (This is called a *change of scale;* it means that one unit along the x axis becomes π/l units along the w axis.)
8. (a) Prove that differentiation switches even functions to odd ones, and odd functions to even ones.
   (b) Prove the same for integration provided that we ignore the constant of integration.
9. Let $\phi(x)$ be a function of period π. If $\phi(x) = \Sigma_{n=1}^{\infty} a_n \sin nx$ for all x, find the odd coefficients.
10. (a) Let $\phi(x)$ be a continuous function on $(0, l)$. Under what conditions is its *odd* extension also a continuous function?
    (b) Let $\phi(x)$ be a differentiable function on $(0, l)$. Under what conditions is its *odd* extension also a differentiable function?
    (c) Same as part (a) for the *even* extension.
    (d) Same as part (b) for the *even* extension.

11. Find the full Fourier series of e^x on $(-l, l)$ in its real and complex forms. (*Hint:* It is convenient to find the complex form first.)
12. Repeat Exercise 11 for $\cosh x$. (*Hint:* Use the preceding result.)
13. Repeat Exercise 11 for $\sin x$. Assume that l is not an integer multiple of π. (*Hint:* First find the series for e^{ix}).
14. Repeat Exercise 11 for $|x|$.
15. Without any computation, predict which of the Fourier coefficients of $|\sin x|$ on the interval $(-\pi, \pi)$ must vanish.
16. Use the De Moivre formulas (11) to derive the standard formulas for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$.
17. Show that a complex-valued function $f(x)$ is real-valued if and only if its complex Fourier coefficients satisfy $c_n = \overline{c_{-n}}$, where $\overline{}$ denotes the complex conjugate.

5.3 ORTHOGONALITY AND GENERAL FOURIER SERIES

Let us try to understand what makes the beautiful method of Fourier series work. For the present let's stick with real functions. If $f(x)$ and $g(x)$ are two real-valued continuous functions defined on an interval $a \le x \le b$, we define their *inner product* to be the integral of their product:

$$(f, g) \equiv \int_a^b f(x)g(x)\,dx. \tag{1}$$

It is a real number. We'll call $f(x)$ and $g(x)$ *orthogonal* if $(f, g) = 0$. (This terminology is supposed to be analogous to the case of ordinary vectors and their inner or dot product.) Notice that no function is orthogonal to itself except $f(x) \equiv 0$. The key observation in each case discussed in Section 5.1 is that **every eigenfunction is orthogonal to every other eigenfunction**. Now we will explain why this fortuitous coincidence is in fact no accident.

We are studying the operator $A = -d^2/dx^2$ with some boundary conditions (either Dirichlet or Neumann or ...). Let $X_1(x)$ and $X_2(x)$ be two different eigenfunctions. Thus

$$\begin{aligned} -X_1'' &= \frac{-d^2X_1}{dx^2} = \lambda_1 X_1 \\ -X_2'' &= \frac{-d^2X_2}{dx^2} = \lambda_2 X_2, \end{aligned} \tag{2}$$

where both functions satisfy the boundary conditions. Let's assume that $\lambda_1 \neq \lambda_2$. We now verify the identity

$$-X_1''X_2 + X_1X_2'' = (-X_1'X_2 + X_1X_2')'.$$

(Work out the right side using the product rule and two of the terms will cancel.) We integrate to get

$$\int_a^b \left(-X_1''X_2 + X_1X_2''\right) dx = \left(-X_1'X_2 + X_1X_2'\right)\Big|_a^b. \tag{3}$$

This is sometimes called *Green's second identity*. If you wished, you could also think of it as the result of two integrations by parts.

On the left side of (3) we now use the differential equations (2). On the right side we use the boundary conditions to reach the following conclusions:

Case 1: Dirichlet. This means that both functions vanish at both ends: $X_1(a) = X_1(b) = X_2(a) = X_2(b) = 0$. So the right side of (3) is zero.

Case 2: Neumann. The first derivatives vanish at both ends. It is once again zero.

Case 3: Periodic. $X_j(a) = X_j(b)$, $X_j'(a) = X_j'(b)$ for both $j = 1, 2$. Again you get zero!

Case 4: Robin. Again you get zero! See Exercise 8.

Thus in all four cases, (3) reduces to

$$(\lambda_1 - \lambda_2)\int_a^b X_1X_2\, dx = 0. \tag{3a}$$

Therefore, X_1 and X_2 are orthogonal! This completely explains why Fourier's method works (at least if $\lambda_1 \neq \lambda_2$)!

The right side of (3) isn't always zero. For example, consider the different boundary conditions: $X(a) = X(b)$, $X'(a) = 2X'(b)$. Then the right side of (3) is $X_1'(b)X_2(b) - X_1(b)X_2'(b)$, which is *not* zero. So the method doesn't always work; the boundary conditions have to be right.

SYMMETRIC BOUNDARY CONDITIONS

So now let us envision *any pair of boundary conditions*

$$\begin{aligned} \alpha_1X(a) + \beta_1X(b) + \gamma_1X'(a) + \delta_1X'(b) = 0 \\ \alpha_2X(a) + \beta_2X(b) + \gamma_2X'(a) + \delta_2X'(b) = 0 \end{aligned} \tag{4}$$

involving eight real constants. (Each of the examples above corresponds to a choice of these constants.) Such a set of boundary conditions is called *symmetric* if

$$\boxed{f'(x)g(x) - f(x)g'(x)\Big|_{x=a}^{x=b} = 0} \tag{5}$$

for any pair of functions $f(x)$ and $g(x)$ both of which satisfy the pair of boundary conditions (4). As we indicated above, each of the four standard boundary conditions (Dirichlet, etc.) is symmetric, but our fifth example is not. The most important thing to keep in mind is that all the standard boundary conditions are symmetric.

Green's second identity (3) then implies the following theorem. By an *eigenfunction* we now mean a solution of $-X'' = \lambda X$ that satisfies (4).

Theorem 1. If you have *symmetric* boundary conditions, then any two eigenfunctions that correspond to distinct eigenvalues are orthogonal. Therefore, if any function is expanded in a series of these eigenfunctions, the coefficients are determined.

Proof. Take two different eigenfunctions $X_1(x)$ and $X_2(x)$ with $\lambda_1 \neq \lambda_2$. We write Green's second identity (3). Because the boundary conditions are symmetric, the right side of (3) vanishes. Because of the different equations, the identity takes the form (3a), and the orthogonality is proven.

If $X_n(x)$ now denotes the eigenfunction with eigenvalue λ_n and if

$$\boxed{\phi(x) = \sum_n A_n X_n(x)} \tag{6}$$

is a convergent series, where the A_n are constants, then

$$(\phi, X_m) = \left(\sum_n A_n X_n, X_m\right) = \sum_n A_n(X_n, X_m) = A_m(X_m, X_m)$$

by the orthogonality. So if we denote $c_m = (X_m, X_m)$, we have

$$\boxed{A_m = \frac{(\phi, X_m)}{c_m}} \tag{7}$$

as the formula for the coefficients. □

Two words of caution. First, we have so far avoided all questions of convergence. Second, if there are two eigenfunctions, say $X_1(x)$ and $X_2(x)$, but their eigenvalues are the same, $\lambda_1 = \lambda_2$, then they don't have to be orthogonal. But if they aren't orthogonal, they can be made so by the Gram–Schmidt orthogonalization procedure (see Exercise 10). For instance, in the case of periodic boundary conditions the two eigenfunctions $\sin(n\pi x/l)$ and $\cos(n\pi x/l)$ are orthogonal on $(-l, l)$, even though they have the same eigenvalue $(n\pi/l)^2$. But the two eigenfunctions $\sin(n\pi x/l)$ and $[\cos(n\pi x/l) + \sin(n\pi x/l)]$ are not orthogonal.

COMPLEX EIGENVALUES

What about complex eigenvalues λ and complex-valued eigenfunctions $X(x)$? If $f(x)$ and $g(x)$ are two complex-valued functions, we define the *inner product* on (a, b) as

$$\boxed{(f, g) = \int_a^b f(x)\overline{g(x)}\,dx.} \tag{8}$$

The bar denotes the complex conjugate. The two functions are called *orthogonal* if $(f, g) = 0$. (This is exactly what is customary for ordinary complex vectors.)

Now suppose that you have the boundary conditions (4) with eight real constants. They are called *symmetric* (or *hermitian*) if

$$f'(x)\,\overline{g(x)} - f(x)\overline{g'(x)}\,\Big|_a^b = 0 \tag{9}$$

for all f, g satisfying the BCs. Then Theorem 1 is true for complex functions without any change at all. But we also have the following important fact.

Theorem 2. Under the same conditions as Theorem 1, *all the eigenvalues are real numbers*. Furthermore, all the eigenfunctions can be chosen to be real valued.

(This could be compared with the discussion at the end of Section 4.1, where complex eigenvalues were discussed explicitly.)

Proof. Let λ be an eigenvalue, possibly complex. Let $X(x)$ be its eigenfunction, also possibly complex. Then $-X'' = \lambda X$ plus BCs. Take the complex conjugate of this equation; thus $-\overline{X}'' = \overline{\lambda}\,\overline{X}$ plus BCs. So $\overline{\lambda}$ is also an eigenvalue. Now use Green's second identity with the functions X and $\overline{X}$. Thus

$$\int_a^b (-X''\overline{X} + X\overline{X}'')\,dx = (-X'\overline{X} + X\overline{X}')\Big|_a^b = 0$$

since the BCs are symmetric. So

$$(\lambda - \overline{\lambda})\int_a^b X\overline{X}\,dx = 0$$

But $X\overline{X} = |X|^2 \geq 0$ and $X(x)$ is not allowed to be the zero function. So the integral *cannot* vanish. Therefore, $\lambda - \overline{\lambda} = 0$, which means exactly that λ *is real*.

Next, let's reconsider the same problem $-X'' = \lambda X$ together with (4), knowing that λ is real. If $X(x)$ is complex, we write it as $X(x) = Y(x) + iZ(x)$, where $Y(x)$ and $Z(x)$ are real. Then $-Y'' - iZ'' = \lambda Y + i\lambda Z$. Equating the real and imaginary parts, we see that $-Y'' = \lambda Y$ and $-Z'' = \lambda Z$. The boundary conditions still hold for both Y and Z because the eight constants in (4) are real numbers. So the *real* eigenvalue λ has the *real* eigenfunctions Y and Z. We could therefore say that X and $\overline{X}$ are replaceable by the Y and Z. The linear combinations $aX + b\overline{X}$ are the same as the linear combinations $cY + dZ$, where a and b are somehow related to c and d. This completes the proof of Theorem 2. □

NEGATIVE EIGENVALUES

We have seen that most of the eigenvalues turn out to be positive. An important question is whether *all* of them are positive. Here is a sufficient condition.

Theorem 3. Assume the same conditions as in Theorem 1. If

$$f(x)f'(x)\Big|_{x=a}^{x=b} \leq 0 \tag{10}$$

for all (real-valued) functions $f(x)$ satisfying the BCs, then there is no negative eigenvalue.

This theorem is proved in Exercise 13. It is easy to verify that (10) is valid for Dirichlet, Neumann, and periodic boundary conditions, so that in these cases there are no negative eigenvalues (see Exercise 11). However, as we have already seen in Section 4.3, it could not be valid for certain Robin boundary conditions.

We have already noticed the close analogy of our analysis with linear algebra. Not only are functions acting as if they were vectors, but the operator $-d^2/dx^2$ is acting like a matrix; in fact, it *is* a linear transformation. Theorems 1 and 2 are like the corresponding theorems about real symmetric matrices. For instance, if A is a real symmetric matrix and f and g are vectors, then $(Af, g) = (f, Ag)$. In our present case, A is a differential operator with symmetric BCs and f and g are functions. The same identity $(Af, g) = (f, Ag)$ holds in our case [see (3)]. The two main differences from matrix theory are, first, that our vector space is infinite dimensional, and second, that the boundary conditions must comprise part of the definition of our linear transformation.

EXERCISES

1. (a) Find the real vectors that are orthogonal to the given vectors [1, 1, 1] and [1, −1, 0].
   (b) Choosing an answer to (a), expand the vector [2, −3, 5] as a linear combination of these three mutually orthogonal vectors.
2. (a) On the interval [−1, 1], show that the function x is orthogonal to the constant functions.
   (b) Find a quadratic polynomial that is orthogonal to both 1 and x.
   (c) Find a cubic polynomial that is orthogonal to all quadratics. (These are the first few *Legendre polynomials*.)

3. Consider $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u(0, t) = 0$, $u_x(l, t) = 0$ and the initial conditions $u(x, 0) = x$, $u_t(x, 0) = 0$. Find the solution explicitly in series form.
4. Consider the problem $u_t = ku_{xx}$ for $0 < x < l$, with the boundary conditions $u(0, t) = U$, $u_x(l, t) = 0$, and the initial condition $u(x, 0) = 0$, where U is a constant.
   (a) Find the solution in series form. (*Hint:* Consider $u(x, t) - U$.)
   (b) Using a direct argument, show that the series converges for $t > 0$.
   (c) If ϵ is a given margin of error, estimate how long a time is required for the value $u(l, t)$ at the endpoint to be approximated by the constant U within the error ϵ. (*Hint:* It is an alternating series with first term U, so that the error is less than the next term.)
5. (a) Show that the boundary conditions $u(0, t) = 0$, $u_x(l, t) = 0$ lead to the eigenfunctions $(\sin(\pi x/2l),\ \sin(3\pi x/2l),\ \sin(5\pi x/2l), \ldots)$.
   (b) If $\phi(x)$ is any function on $(0, l)$, derive the expansion
   $$\phi(x) = \sum_{n=0}^{\infty} C_n \sin\left\{\left(n + \frac{1}{2}\right)\frac{\pi x}{l}\right\} \quad (0 < x < l)$$
   by the following method. Extend $\phi(x)$ to the function $\tilde{\phi}$ defined by $\tilde{\phi}(x) = \phi(x)$ for $0 \le x \le l$ and $\tilde{\phi}(x) = \phi(2l - x)$ for $l \le x \le 2l$. (This means that you are extending it *evenly across* $x = l$.) Write the Fourier sine series for $\tilde{\phi}(x)$ on the interval $(0, 2l)$ and write the formula for the coefficients.
   (c) Show that every second coefficient vanishes.
   (d) Rewrite the formula for C_n as an integral of the original function $\phi(x)$ on the interval $(0, l)$.
6. Find the complex eigenvalues of the first-derivative operator d/dx subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0, 1)$?
7. Show *by direct integration* that the eigenfunctions associated with the Robin BCs, namely,
   $$\phi_n(x) = \cos\beta_n x + \frac{a_0}{\beta_n}\sin\beta_n x \quad \text{where } \lambda_n = \beta_n^2,$$
   are mutually orthogonal on $0 \le x \le l$, where β_n are the positive roots of (4.3.8).
8. Show directly that $(-X_1' X_2 + X_1 X_2')|_a^b = 0$ if both X_1 and X_2 satisfy the same Robin boundary condition at $x = a$ and the same Robin boundary condition at $x = b$.
9. Show that the boundary conditions
   $$X(b) = \alpha X(a) + \beta X'(a) \quad \text{and} \quad X'(b) = \gamma X(a) + \delta X'(a)$$
   on an interval $a \le x \le b$ are symmetric if and only if $\alpha\delta - \beta\gamma = 1$.
10. (*The Gram–Schmidt orthogonalization procedure*) If $X_1, X_2, \ldots$ is any sequence (finite or infinite) of linearly independent vectors in any vector

space with an inner product, it can be replaced by a sequence of linear combinations that are mutually orthogonal. The idea is that at each step one subtracts off the components parallel to the previous vectors. The procedure is as follows. First, we let $Z_1 = X_1/\|X_1\|$. Second, we define

$$Y_2 = X_2 - (X_2, Z_1)Z_1 \quad \text{and} \quad Z_2 = \frac{Y_2}{\|Y_2\|}.$$

Third, we define

$$Y_3 = X_3 - (X_3, Z_2)Z_2 - (X_3, Z_1)Z_1 \quad \text{and} \quad Z_3 = \frac{Y_3}{\|Y_3\|},$$

and so on.

(a) Show that all the vectors $Z_1, Z_2, Z_3, \ldots$ are orthogonal to each other.

(b) Apply the procedure to the pair of functions $\cos x + \cos 2x$ and $3\cos x - 4\cos 2x$ in the interval $(0, \pi)$ to get an orthogonal pair.

11. (a) Show that the condition $f(x)f'(x)\big|_a^b \le 0$ is valid for any function $f(x)$ that satisfies Dirichlet, Neumann, or periodic boundary conditions.

    (b) Show that it is also valid for Robin BCs provided that the constants a_0 and a_l are positive.

12. Prove *Green's first identity:* For every pair of functions $f(x)$, $g(x)$ on (a, b),

$$\int_a^b f''(x)g(x)\,dx = -\int_a^b f'(x)g'(x)\,dx + f'g\bigg|_a^b.$$

13. Use Green's first identity to prove Theorem 3. (*Hint:* Substitute $f(x) = X(x) = g(x)$, a real eigenfunction.)

14. What do the terms in the series

$$\frac{\pi}{4} = \sin 1 + \frac{1}{3}\sin 3 + \frac{1}{5}\sin 5 + \cdots$$

look like? Make a graph of $\sin n$ for $n = 1, 2, 3, 4, \ldots, 20$ without drawing the intervening curve; that is, just plot the 20 points. Use a calculator; remember that we are using radians. In some sense the numbers $\sin n$ are *randomly* located in the interval $(-1, 1)$. There is a great deal of "random cancellation" in the series.

15. Use the same idea as in Exercises 12 and 13 to show that none of the eigenvalues of the fourth-order operator $+d^4/dx^4$ with the boundary conditions $X(0) = X(l) = X''(0) = X''(l) = 0$ are negative. □

5.4 COMPLETENESS

In this section we state the basic theorems about the convergence of Fourier series. We discuss three senses of convergence of functions. The basic theorems

(Theorems 2, 3, and 4) state sufficient conditions on a function $f(x)$ that its Fourier series converge to it in these three senses. Most of the proofs are difficult, however, and we omit them for now. At the end of the section we discuss the mean-square convergence in greater detail and use it to define the notion of completeness.

Consider the eigenvalue problem

$$X'' + \lambda X = 0 \text{ in } (a, b) \text{ with any } \textit{symmetric} \text{ BC.} \tag{1}$$

By Theorem 5.3.2, we know that all the eigenvalues λ are real.

Theorem 1. There are an infinite number of eigenvalues. They form a sequence $\lambda_n \to +\infty$.

For a proof of Theorem 1, see Chapter 11 or [CL]. We may assume that the eigenfunctions $X_n(x)$ are pairwise orthogonal and real valued (see Section 5.3). For instance, if k linearly independent eigenfunctions correspond to the *same* eigenvalue λ_n, then they can be rechosen to be orthogonal and real, and the sequence may be numbered so that λ_n is repeated k times. Thus we may list the eigenvalues as

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \to +\infty \tag{2}$$

with the corresponding eigenfunctions

$$X_1, X_2, X_3, \ldots, \tag{3}$$

which are pairwise orthogonal. Some interesting examples were found in Section 4.3.

For any function $f(x)$ on (a, b), its *Fourier coefficients* are defined as

$$\boxed{A_n = \frac{(f, X_n)}{(X_n, X_n)} = \frac{\int_a^b f(x)\overline{X_n(x)}\,dx}{\int_a^b |X_n(x)|^2\,dx}.} \tag{4}$$

Its *Fourier series* is the series $\Sigma_n A_n X_n(x)$.

In this section we present three convergence theorems. Just to convince the skeptic that convergence theorems are more than a pedantic exercise, we mention the curious fact that *there exists an integrable function $f(x)$ whose Fourier series diverges at every point x*! There even exists a *continuous* function whose Fourier series diverges at many points! See [Zy] for proofs.

To set the stage we need to introduce various notions of convergence. This is a good point for the reader to review the basic facts about infinite series (outlined in Section A.2).

THREE NOTIONS OF CONVERGENCE

Definition. We say that an infinite series $\Sigma_{n=1}^{\infty} f_n(x)$ *converges* to $f(x)$ *pointwise* in (a, b) if it converges to $f(x)$ for *each* $a < x < b$. That is, for each

$a < x < b$ we have

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as } N \to \infty. \tag{5}$$

Definition. We say that the series *converges uniformly* to $f(x)$ in $[a, b]$ if

$$\max_{a \le x \le b} \left| f(x) - \sum_{n=1}^{N} f_n(x) \right| \to 0 \quad \text{as } N \to \infty. \tag{6}$$

(Note that the endpoints are included in this definition.) That is, you take the biggest difference over all the x's and *then* take the limit.

The two preceding concepts of convergence are also discussed in Section A.2. A third important concept is the following one.

Definition. We say the series *converges in the mean-square (or L^2) sense* to $f(x)$ in (a, b) if

$$\int_a^b \left| f(x) - \sum_{n=1}^{N} f_n(x) \right|^2 dx \to 0 \quad \text{as } N \to \infty. \tag{7}$$

Thus we take the integral instead of the maximum. (The terminology L^2 refers to the square inside the integral.)

Notice that uniform convergence is stronger than both pointwise and L^2 convergence (see Exercise 2.) Figure 1 illustrates a typical uniformly convergent series by graphing both $f(x)$ and a partial sum for large N.

Example 1.

Let $f_n(x) = (1 - x)x^{n-1}$ on the interval $0 < x < 1$. Then the series is "telescoping." The partial sums are

$$\sum_{n=1}^{N} f_n(x) = \sum_{1}^{N} (x^{n-1} - x^n) = 1 - x^N \to 1 \quad \text{as } N \to \infty$$

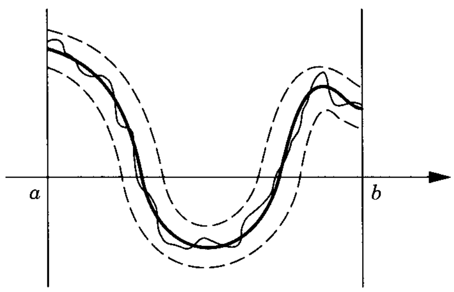

Figure 1

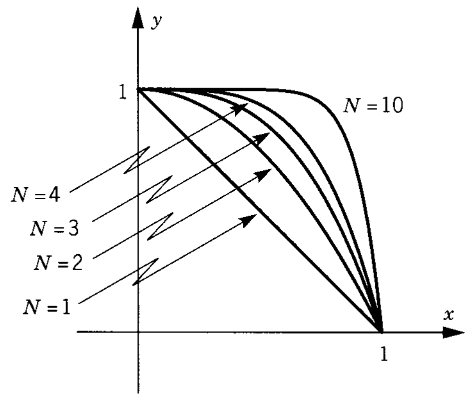

Figure 2

because $x < 1$. This convergence is valid for *each* x. Thus $\sum_{n=1}^{\infty} f_n(x) = 1$ pointwise. In words, the series *converges pointwise* to the function $f(x) \equiv 1$.

But the convergence is *not uniform* because $\max\,[1 - (1 - x^N)] = \max x^N = 1$ for every N. However, it *does converge in mean-square* since

$$\int_0^1 \left|x^N\right|^2 dx = \frac{1}{2N+1} \to 0.$$

Figure 2 is a sketch of a few partial sums of Example 1. □

Example 2.

Let

$$f_n(x) = \frac{n}{1 + n^2x^2} - \frac{n-1}{1 + (n-1)^2x^2}$$

in the interval $0 < x < l$. This series also telescopes so that

$$\sum_{n=1}^{N} f_n(x) = \frac{N}{1 + N^2x^2} = \frac{1}{N[(1/N^2) + x^2]} \to 0 \text{ as } N \to \infty \text{ if } x > 0.$$

So the series *converges pointwise* to the sum $f(x) \equiv 0$.

On the other hand,

$$\int_0^l \left[\sum_{n=1}^{N} f_n(x)\right]^2 dx = \int_0^l \frac{N^2}{(1 + N^2x^2)^2}\, dx$$
$$= N \int_0^{Nl} \frac{1}{(1 + y^2)^2}\, dy \to +\infty \quad (\text{where } y = Nx)$$

because

$$\int_0^{Nl} \frac{1}{(1+y^2)^2}\,dy \to \int_0^{\infty} \frac{1}{(1+y^2)^2}\,dy.$$

So the series *does not converge in the mean-square sense*. Also, it *does not converge uniformly* because

$$\max_{(0,l)} \frac{1}{1+N^2x^2} = N,$$

which obviously does not tend to zero as $N \to \infty$. □

CONVERGENCE THEOREMS

Now let $f(x)$ be any function defined on $a \le x \le b$. Consider the Fourier series for the problem (1) with any given boundary conditions that are symmetric. We now state a convergence theorem for each of the three modes of convergence. They are partly proved in the next section.

Theorem 2. Uniform Convergence The Fourier series $\Sigma A_n X_n(x)$ converges to $f(x)$ uniformly on $[a, b]$ provided that

(i) $f(x), f'(x)$, and $f''(x)$ exist and are continuous for $a \le x \le b$ and

(ii) $f(x)$ satisfies the given boundary conditions.

Theorem 2 assures us of a very good kind of convergence provided that the conditions on $f(x)$ and its derivatives are met. For the classical Fourier series (full, sine, and cosine), it is not required that $f''(x)$ exist.

Theorem 3. L^2 Convergence The Fourier series converges to $f(x)$ in the mean-square sense in (a, b) provided only that $f(x)$ is any function for which

$$\int_a^b |f(x)|^2\,dx \text{ is finite.} \tag{8}$$

Theorem 3 assures us of a certain kind of convergence under a very weak assumption on $f(x)$. [We could even use the very general Lebesgue integral here instead of the standard (Riemann) integral encountered in calculus courses. In fact, the Lebesgue integral was invented in order that Theorem 3 be true for the most general possible functions.]

Third, we present a theorem that is intermediate as regards the hypotheses on $f(x)$. It requires two more definitions. A function $f(x)$ has a *jump discontinuity* at a point $x = c$ if the one-sided limits $f(c+)$ and $f(c-)$ exist but are not equal. [It doesn't matter what $f(c)$ happens to be or even whether $f(c)$ is defined or not.] The *value* of the jump discontinuity is the number $f(c+) - f(c-)$. See Figure 3 for a function with two jumps.

A function $f(x)$ is called *piecewise continuous* on an interval $[a, b]$ if it is continuous at all but a finite number of points and has jump discontinuities

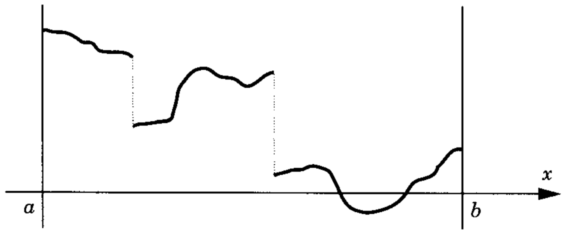

Figure 3

at those points. Another way to say this is that at every point in the interval (including the endpoints) the one-sided limits $f(c+)$ and $f(c-)$ exist; and except at a finite number of points they are equal. For these definitions, see also Section A.1. A typical piecewise continuous function is sketched in Figure 3. The function $Q(x, 0)$ in Section 2.4 is an example of a piecewise continuous function.

Theorem 4. Pointwise Convergence of Classical Fourier Series

(i) The classical Fourier series (full or sine or cosine) converges to $f(x)$ pointwise on (a, b) provided that $f(x)$ is a continuous function on $a \le x \le b$ and $f'(x)$ is piecewise continuous on $a \le x \le b$.

(ii) More generally, if $f(x)$ itself is only piecewise continuous on $a \le x \le b$ and $f'(x)$ is also piecewise continuous on $a \le x \le b$, then the classical Fourier series converges at every point $x(-\infty < x < \infty)$. The sum is

$$\sum_n A_n X_n(x) = \tfrac{1}{2}\,[f(x+) + f(x-)] \qquad \text{for all } a < x < b. \quad (9)$$

The sum is $\frac{1}{2}[f_{\text{ext}}(x+) + f_{\text{ext}}(x-)]$ for all $-\infty < x < \infty$, where $f_{\text{ext}}(x)$ is the extended function (periodic, odd periodic, or even periodic).

Thus at a jump discontinuity the series converges to the *average* of the limits from the right and from the left. In the case of the Fourier sine (or cosine) series on $(0, l)$, the extended function $f_{\text{ext}}(x)$ is the odd (or even) function of period $2l$. For the full series on $(-l, l)$, it is the periodic extension. The extension is piecewise continuous with a piecewise continuous derivative on $(-\infty, \infty)$.

It is convenient to restate Theorem 4 directly for functions that are already defined on the whole line. By considering the periodic, even, and odd extensions of functions, Theorem 4 is equivalent to the following statement.

Theorem 4∞. If $f(x)$ is a function of period $2l$ on the line for which $f(x)$ and $f'(x)$ are piecewise continuous, then the classical full Fourier series converges to $\frac{1}{2}[f(x+) + f(x-)]$ for $-\infty < x < \infty$.

The Fourier series of a continuous but nondifferentiable function $f(x)$ is not guaranteed to converge pointwise. By Theorem 3 it must converge to $f(x)$ in the L^2 sense. If we wanted to be sure of its pointwise convergence, we would have to know something about its derivative $f'(x)$.

Example 3.

The Fourier sine series of the function $f(x) \equiv 1$ on the interval $(0, \pi)$ is

$$\sum_{n \text{ odd}} \frac{4}{n\pi} \sin nx. \tag{10}$$

Although it converges at each point, this series does not converge uniformly on $[0, \pi]$. One reason is that the series equals zero at both endpoints (0 and π) but the function is 1 there. Condition (ii) of Theorem 2 is not satisfied: the boundary conditions are Dirichlet and the function $f(x)$ does not vanish at the endpoints. However, Theorem 4(i) is applicable, so that the series does converge pointwise to $f(x)$. Thus (10) must sum to 1 for every $0 < x < \pi$. For instance, we get a true equation if we put $x = \pi/2$:

$$1 = f\left(\frac{\pi}{2}\right) = \sum_{n \text{ odd}} \frac{4}{n\pi}(-1)^{(n-1)/2} = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}.$$

Therefore, we get the convergent series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots .$$

Noting that $0 < 1 < \pi$, we may put $x = 1$ to get the convergent series

$$\frac{\pi}{4} = \sin 1 + \frac{1}{3}\sin 3 + \frac{1}{5}\sin 5 + \cdots .$$

Other amusing series are obtainable in this way. □

Another important question, especially for our purposes, is whether a Fourier series can be *differentiated* term by term. Take the case of (10). On the left side the derivative is zero. On the right side we ought to get the series.

$$\frac{4}{\pi}\sum_{n \text{ odd}} \cos nx. \tag{11}$$

But this is clearly divergent because the terms don't even tend to zero as $n \to \infty$ (the nth *term test* for divergence)! So in this example you *cannot differentiate term by term*. For a more general viewpoint, however, see Example 8 in Section 12.1.

Differentiation of a Fourier series is a delicate matter. But integration term by term is not delicate and is usually valid (see Exercise 11).

The proofs of Theorems 1 to 4 are lengthy and will be postponed to the next section and to Chapter 11. For complete proofs of Theorems 2 and 3, see Section 7.4 of [CL]. For complete proofs of the classical cases of Theorems 2, 3, and 4, see [DM] or [CH]. Of the three convergence theorems, Theorem 3 is the easiest one to apply because $f'(x)$ does not have to exist and $f(x)$ itself does not even have to be continuous. We now pursue a set of ideas that is related to Theorem 3 and is important in quantum mechanics.

THE L^2 THEORY

The main idea is to regard orthogonality as if it were a geometric property. We have already defined the inner product on (a, b) as

$$(f, g) = \int_a^b f(x)\overline{g(x)}\,dx.$$

[In case the functions are real valued, we just ignore the complex conjugate $(\overline{\ })$.] We now define the L^2 *norm of* f as

$$\|f\| = (f, f)^{1/2} = \left[\int_a^b |f(x)|^2\,dx\right]^{1/2}.$$

The quantity

$$\|f - g\| = \left[\int_a^b |f(x) - g(x)|^2\,dx\right]^{1/2} \tag{12}$$

is a measurement of the "distance" between two functions f and g. It is sometimes called the L^2 *metric*. The concept of a metric was first mentioned in Section 1.5; the L^2 metric is the nicest one.

Theorem 3 can be restated as follows. If $\{X_n\}$ are the eigenfunctions associated with a set of symmetric BCs and if $\|f\| < \infty$, then

$$\left\| f - \sum_{n \le N} A_n X_n \right\| \to 0 \qquad \text{as } N \to \infty. \tag{13}$$

That is, the partial sums get nearer and nearer to f.

Theorem 5. Least-Square Approximation Let $\{X_n\}$ be any orthogonal set of functions. Let $\|f\| < \infty$. Let N be a *fixed* positive integer. Among all possible choices of N constants $c_1, c_2, \ldots, c_N$, the choice that minimizes

$$\boxed{\left\| f - \sum_{n=1}^{N} c_n X_n \right\|}$$

is $c_1 = A_1, \ldots, c_n = A_n$.

(These are the Fourier coefficients! It means that the linear combination of $X_1, \ldots, X_n$ which approximates f most closely is the Fourier combination!)

Proof. For the sake of simplicity we assume in this proof that $f(x)$ and all the $X_n(x)$ are real valued. Denote the error (remainder) by

$$E_N = \left\| f - \sum_{n\le N} c_n X_n \right\|^2 = \int_a^b \left| f(x) - \sum_{n\le N} c_n X_n(x) \right|^2 dx. \tag{14}$$

Expanding the square, we have (assuming the functions are real valued)

$$E_N = \int_a^b |f(x)|^2\, dx - 2\sum_{n\le N} c_n \int_a^b f(x)X_n(x)\,dx$$
$$+ \sum_n \sum_m c_n c_m \int_a^b X_n(x)X_m(x)\,dx.$$

Because of orthogonality, the last integral vanishes except for $n = m$. So the double sum reduces to $\Sigma c_n^2 \int |X_n|^2\, dx$. Let us write this in the norm notation:

$$E_N = \|f\|^2 - 2\sum_{n\le N} c_n(f, X_n) + \sum_{n\le N} c_n^2 \|X_n\|^2 .$$

We may "complete the square":

$$E_N = \sum_{n\le N} \|X_n\|^2 \left[c_n - \frac{(f, X_n)}{\|X_n\|^2} \right]^2 + \|f\|^2 - \sum_{n\le N} \frac{(f, X_n)^2}{\|X_n\|^2}. \tag{15}$$

Now the coefficients c_n appear in only one place, inside the squared term. The expression is clearly smallest if the squared term vanishes. That is,

$$c_n = \frac{(f, X_n)}{\|X_n\|^2} \equiv A_n,$$

which proves Theorem 5. □

The completion of the square has further consequences. Let's choose the c_n to be the Fourier coefficients: $c_n = A_n$. The last expression (15) for the error E_N becomes

$$0 \le E_N = \|f\|^2 - \sum_{n\le N} \frac{(f, X_n)^2}{\|X_n\|^2} = \|f\|^2 - \sum_{n\le N} A_n^2 \|X_n\|^2. \tag{16}$$

Because this is positive, we have

$$\sum_{n\le N} A_n^2 \int_a^b |X_n(x)|^2\, dx \le \int_a^b |f(x)|^2\, dx. \tag{17}$$

On the left side we have the partial sums of a series of positive terms with bounded partial sums. Therefore, the corresponding infinite series converges and its sum satisfies

$$\boxed{\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2\, dx \le \int_a^b |f(x)|^2\, dx.} \tag{18}$$

This is known as *Bessel's inequality*. It is valid as long as the integral of $|f|^2$ is finite.

Theorem 6. The Fourier series of $f(x)$ converges to $f(x)$ in the mean-square sense if and only if

$$\boxed{\sum_{n=1}^{\infty} |A_n|^2 \int_a^b |X_n(x)|^2\,dx = \int_a^b |f(x)|^2\,dx} \tag{19}$$

(i.e., if and only if you have equality).

Proof. Mean-square convergence means that the remainder $E_N \to 0$. But from (16) this means that $\Sigma_{n\le N}|A_n|^2\|X_n\|^2 \to \|f\|^2$, which in turn means (19), known as *Parseval's equality*.

Definition. The infinite orthogonal set of functions $\{X_1(x), X_2(x), \ldots\}$ is called *complete* if Parseval's equality (19) is true for all f with $\int_a^b |f|^2\,dx < \infty$.

Theorem 3 asserts that the set of eigenfunctions coming from (1) is always complete. Thus we have the following conclusion.

Corollary 7. If $\int_a^b |f(x)|^2 dx$ is finite, then the Parseval equality (19) is true.

Example 4.

Consider once again the Fourier series (10). Parseval's equality asserts that

$$\sum_{n\ \text{odd}} \left(\frac{4}{n\pi}\right)^2 \int_0^\pi \sin^2 nx\,dx = \int_0^\pi 1^2\,dx.$$

This means that

$$\sum_{n\ \text{odd}} \left(\frac{4}{n\pi}\right)^2 \frac{\pi}{2} = \pi.$$

In other words,

$$\sum_{n\ \text{odd}} \frac{1}{n^2} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots = \frac{\pi^2}{8},$$

another interesting numerical series. □

For a full discussion of completeness using the concept of the Lebesgue integral, see [LL] for instance.

EXERCISES

1. $\Sigma_{n=0}^{\infty}(-1)^n x^{2n}$ is a geometric series.
   (a) Does it converge pointwise in the interval $-1 < x < 1$?
   (b) Does it converge uniformly in the interval $-1 < x < 1$?
   (c) Does it converge in the L^2 sense in the interval $-1 < x < 1$? (*Hint:* You can compute its partial sums explicitly.)
2. Consider any series of functions on any finite interval. Show that if it converges uniformly, then it also converges in the L^2 sense and in the pointwise sense.
3. Let γ_n be a sequence of constants tending to ∞. Let $f_n(x)$ be the sequence of functions defined as follows: $f_n\left(\frac{1}{2}\right) = 0$, $f_n(x) = \gamma_n$ in the interval $[\frac{1}{2} - \frac{1}{n}, \frac{1}{2})$, let $f_n(x) = -\gamma_n$ in the interval $(\frac{1}{2}, \frac{1}{2} + \frac{1}{n}]$ and let $f_n(x) = 0$ elsewhere. Show that:
   (a) $f_n(x) \to 0$ pointwise.
   (b) The convergence is not uniform.
   (c) $f_n(x) \to 0$ in the L^2 sense if $\gamma_n = n^{1/3}$.
   (d) $f_n(x)$ does not converge in the L^2 sense if $\gamma_n = n$.
4. Let

$$g_n(x) = \begin{cases} 1 \text{ in the interval } \left[\dfrac{1}{4} - \dfrac{1}{n^2}, \dfrac{1}{4} + \dfrac{1}{n^2}\right) & \text{for odd } n \\ 1 \text{ in the interval } \left[\dfrac{3}{4} - \dfrac{1}{n^2}, \dfrac{3}{4} + \dfrac{1}{n^2}\right) & \text{for even } n \\ 0 & \text{for all other } x. \end{cases}$$

   Show that $g_n(x) \to 0$ in the L^2 sense but that $g_n(x)$ does not tend to zero in the pointwise sense.
5. Let $\phi(x) = 0$ for $0 < x < 1$ and $\phi(x) = 1$ for $1 < x < 3$.
   (a) Find the first four nonzero terms of its Fourier cosine series explicitly.
   (b) For each x $(0 \le x \le 3)$, what is the sum of this series?
   (c) Does it converge to $\phi(x)$ in the L^2 sense? Why?
   (d) Put $x = 0$ to find the sum

$$1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5} + \frac{1}{7} + \frac{1}{8} - \frac{1}{10} - \frac{1}{11} + \cdots.$$

6. Find the sine series of the function $\cos x$ on the interval $(0, \pi)$. For each x satisfying $-\pi \le x \le \pi$, what is the sum of the series?
7. Let

$$\phi(x) = \begin{cases} -1 - x & \text{for } -1 < x < 0 \\ +1 - x & \text{for } 0 < x < 1. \end{cases}$$

(a) Find the full Fourier series of $\phi(x)$ in the interval $(-1, 1)$.
(b) Find the first three nonzero terms explicitly.
(c) Does it converge in the mean square sense?
(d) Does it converge pointwise?
(e) Does it converge uniformly to $\phi(x)$ in the interval $(-1, 1)$?

8. Consider the Fourier sine series of each of the following functions. In this exercise do not compute the coefficients but use the general convergence theorems (Theorems 2, 3, and 4) to discuss the convergence of each of the series in the pointwise, uniform, and L^2 senses.
(a) $f(x) = x^3$ on $(0, l)$.
(b) $f(x) = lx - x^2$ on $(0, l)$.
(c) $f(x) = x^{-2}$ on $(0, l)$.

9. Let $f(x)$ be a function on $(-l, l)$ that has a continuous derivative and satisfies the periodic BCs. Let a_n and b_n be the Fourier coefficients of $f(x)$, and let a'_n and b'_n be the Fourier coefficients of its derivative $f'(x)$. Show that

$$a'_n = \frac{n\pi b_n}{l} \qquad \text{and} \qquad b'_n = \frac{-n\pi a_n}{l} \qquad \text{for } n \neq 0.$$

(*Hint:* Write the formulas for a'_n and b'_n and integrate by parts.) This means that the Fourier series of $f'(x)$ is what you'd obtain as if you differentiated term by term. It does not mean that the differentiated series converges.

10. Deduce from Exercise 9 that there is a constant k so that

$$|a_n| + |b_n| \leq \frac{k}{n} \qquad \text{for all } n.$$

11. (*Term by term integration*)
(a) If $f(x)$ is a piecewise continuous function in $[-l, l]$, show that its indefinite integral $F(x) = \int_{-l}^{x} f(s)\,ds$ has a full Fourier series that converges pointwise.
(b) Write this convergent series for $f(x)$ explicitly in terms of the Fourier coefficients a_0, a_n, b_n of $f(x)$.
(*Hint:* Apply a convergence theorem. Write the formulas for the coefficients and integrate by parts.)

12. Start with the Fourier sine series of $f(x) = x$ on the interval $(0, l)$. Apply Parseval's equality. Find the sum $\Sigma_{n=1}^{\infty} 1/n^2$.

13. Start with the Fourier cosine series of $f(x) = x^2$ on the interval $(0, l)$. Apply Parseval's equality. Find the sum $\Sigma_{n=1}^{\infty} 1/n^4$.

14. Find the sum $\Sigma_{n=1}^{\infty} 1/n^6$.

15. Let $\phi(x) \equiv 1$ for $0 < x < \pi$. Expand

$$1 = \sum_{n=0}^{\infty} B_n \cos\left[\left(n + \tfrac{1}{2}\right) x\right].$$

(a) Find B_n.
(b) Let $-2\pi < x < 2\pi$. For which such x does this series converge? For each such x, what is the sum of the series? [*Hint:* Think of extending $\phi(x)$ beyond the interval $(0, \pi)$.]
(c) Apply Parseval's equality to this series. Use it to calculate the sum

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots .$$

16. Let $\phi(x) = |x|$ in $(-\pi, \pi)$. If we approximate it by the function

$$f(x) = \tfrac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x,$$

what choice of coefficients will minimize the L^2 error?
17. Modify the proofs of Theorems 5 and 6 for the case of complex-valued functions.
18. Consider a solution of the wave equation with $c = 1$ on $[0, l]$ with homogeneous Dirichlet or Neumann boundary conditions.
(a) Show that its energy $E = \frac{1}{2}\int_0^l (u_t^2 + u_x^2)\,dx$ is a constant.
(b) Let $E_n(t)$ be the energy of its nth harmonic (the nth term in the expansion). Show that $E = \Sigma E_n$. (*Hint:* Use the orthogonality. Assume that you can integrate term by term.)
19. Here is a general method to calculate the *normalizing constants*. Let $X(x, \lambda)$ be a family of real solutions of the ODE $-X'' = \lambda X$ which depends in a smooth manner on λ as well as on x.
(a) Find the ODE satisfied by $\partial X/\partial\lambda$.
(b) Apply Green's second identity to the pair of functions X and $\partial X/\partial\lambda$ in order to obtain a formula for $\int_a^b X^2 dx$ in terms of the boundary values.
(c) As an example, use the result of part (b) and the Dirichlet boundary conditions to compute $\int_0^l \sin^2(m\pi x/l)\,dx$.
20. Use the method of Exercise 19 to compute the normalizing constants $\int_0^l X^2\,dx$ in the case of the Robin boundary conditions.

5.5 COMPLETENESS AND THE GIBBS PHENOMENON

Our purpose here is to *prove* the pointwise convergence of the classical full Fourier series. This will lead to the celebrated Gibbs phenomenon for jump discontinuities.

We may as well take the whole-line case, Theorem 4∞ of Section 5.4. To avoid technicalities, let us begin with a C^1 function $f(x)$ on the whole line of period $2l$. (A C^1 function is a function that has a continuous derivative in $(-\infty, \infty)$; see Section A.1.) We also assume that $l = \pi$, which can easily be arranged through a change of scale (see Exercise 5.2.7).

Thus the Fourier series is

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos nx + B_n \sin nx) \tag{1}$$

with the coefficients

$$A_n = \int_{-\pi}^{\pi} f(y) \cos ny \frac{dy}{\pi} \qquad (n = 0, 1, 2, \ldots)$$

$$B_n = \int_{-\pi}^{\pi} f(y) \sin ny \frac{dy}{\pi} \qquad (n = 1, 2, \ldots).$$

The Nth partial sum of the series is

$$S_N(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx). \tag{2}$$

We want to prove that $S_N(x)$ converges to $f(x)$ as $N \to \infty$. Pointwise convergence means that x is kept fixed as we take the limit.

The first step of the proof is to stick the formulas for the coefficients into the partial sum and rearrange the terms. Doing this, we get

$$S_N(x) = \int_{-\pi}^{\pi} \left[1 + 2\sum_{n=1}^{N} (\cos ny \cos nx + \sin ny \sin nx) \right] f(y) \frac{dy}{2\pi}.$$

Inside the parentheses is the cosine of a difference of angles, so we can summarize the formula as

$$S_N(x) = \int_{-\pi}^{\pi} K_N(x - y) f(y) \frac{dy}{2\pi}, \tag{3}$$

where

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos n\theta. \tag{4}$$

The second step is to study the properties of this function, called the *Dirichlet kernel*. Notice that $K_N(\theta)$ has period 2π and that

$$\int_{-\pi}^{\pi} K_N(\theta) \frac{d\theta}{2\pi} = 1 + 0 + 0 + \cdots + 0 = 1.$$

It is a remarkable fact that the series for K_N can be summed! In fact,

$$\boxed{K_N(\theta) = \frac{\sin\left(N + \frac{1}{2}\right)\theta}{\sin \frac{1}{2}\theta}.} \tag{5}$$

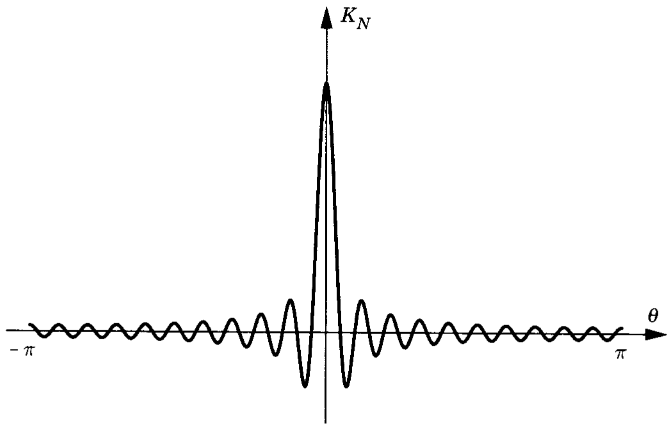

Figure 1

Proof of (5). The easiest proof is by complexification. By De Moivre's formula for complex exponentials,

$$K_N(\theta) = 1 + \sum_{n=1}^{N} (e^{in\theta} + e^{-in\theta}) = \sum_{n=-N}^{N} e^{in\theta}$$
$$= e^{-iN\theta} + \cdots + 1 + \cdots + e^{iN\theta}.$$

This is a finite geometric series with the first term $e^{-iN\theta}$, the ratio $e^{i\theta}$, and the last term $e^{iN\theta}$. So it adds up to

$$K_N(\theta) = \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}}$$

$$= \frac{e^{-i\left(N+\frac{1}{2}\right)\theta} - e^{+i\left(N+\frac{1}{2}\right)\theta}}{-e^{\frac{1}{2}i\theta} + e^{-\frac{1}{2}i\theta}}$$

$$= \frac{\sin\left[\left(N+\frac{1}{2}\right)\theta\right]}{\sin\frac{1}{2}\theta}. \qquad \square$$

Figure 1 is a sketch of $K_N(\theta)$. (It looks somewhat like the diffusion kernel, the source function of Section 2.4, except for its oscillatory tail.)

The third step is to combine (3) with (5). Letting $\theta = y - x$ and using the evenness of K_N, formula (3) takes the form

$$S_N(x) = \int_{-\pi}^{\pi} K_N(\theta) f(x+\theta) \frac{d\theta}{2\pi}.$$

The interval of integration really ought to be $[x-\pi, x+\pi]$, but since both K_N and f have period 2π, any interval of length 2π will do. Next we subtract the constant $f(x) = f(x) \cdot 1$ and use formula (5) to get

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} K_N(\theta)\,[f(x+\theta) - f(x)]\frac{d\theta}{2\pi}$$

or

$$S_N(x) - f(x) = \int_{-\pi}^{\pi} g(\theta)\sin\left[\left(N + \tfrac{1}{2}\right)\theta\right]\frac{d\theta}{2\pi}, \tag{6}$$

where

$$g(\theta) = \frac{f(x+\theta) - f(x)}{\sin\frac{1}{2}\theta} \tag{7}$$

Remember that x remains fixed. All we have to show is that the integral (6) tends to zero as $N \to \infty$.

That is the fourth step. We notice that the functions

$$\phi_N(\theta) = \sin\left[\left(N + \tfrac{1}{2}\right)\theta\right] \qquad (N = 1, 2, 3, \ldots) \tag{8}$$

form an *orthogonal* set on the interval $(0, \pi)$ because they correspond to mixed boundary conditions (see Exercise 5.3.5). Hence they are also orthogonal on the interval $(-\pi, \pi)$. Therefore, *Bessel's inequality* (5.4.18) is valid:

$$\sum_{N=1}^{\infty} \frac{|(g, \phi_N)|^2}{\|\phi_N\|^2} \le \|g\|^2. \tag{9}$$

By direct calculation, $\|\phi_N\|^2 = \pi$. If $\|g\| < \infty$, the series (9) is convergent and its terms tend to zero. So $(g, \phi_N) \to 0$, which says exactly that the integral in (6) tends to zero.

The final step is to check that $\|g\| < \infty$. We have

$$\|g\|^2 = \int_{-\pi}^{\pi} \frac{[f(x+\theta) - f(x)]^2}{\sin^2\frac{1}{2}\theta}\,d\theta.$$

Since the numerator is continuous, the only possible difficulty could occur where the sine vanishes, namely at $\theta = 0$. At that point,

$$\lim_{\theta\to 0} g(\theta) = \lim_{\theta\to 0} \frac{f(x+\theta) - f(x)}{\theta} \cdot \frac{\theta}{\sin\frac{1}{2}\theta} = 2f'(x) \tag{11}$$

by L'Hôpital's rule [since $f(x)$ is differentiable]. Therefore, $g(\theta)$ is everywhere continuous, so that the integral $\|g\|$ is finite. This completes the proof of pointwise convergence of the Fourier series of any C^1 function. □

PROOF FOR DISCONTINUOUS FUNCTIONS

If the periodic function $f(x)$ itself is only piecewise continuous and $f'(x)$ is also piecewise continuous on $-\infty < x < \infty$, we want to prove that the Fourier series converges and that its sum is $\frac{1}{2}[f(x+) + f(x-)]$ (see

Theorem 5.4.4∞). This means that we assume that $f(x)$ and $f'(x)$ are continuous except at a finite number of points, and at those points they have jump discontinuities.

The proof begins as before. However, we modify the third step, replacing (6) by

$$\begin{aligned} S_N(x) - \frac{1}{2}[f(x+) + f(x-)] &= \int_0^{\pi} K_N(\theta)[f(x+\theta) - f(x+)]\frac{d\theta}{2\pi} \\ &\quad + \int_{-\pi}^{0} K_N(\theta)[f(x+\theta) - f(x-)]\frac{d\theta}{2\pi} \\ &= \int_0^{\pi} g_+(\theta)\sin\left[\left(N+\tfrac{1}{2}\right)\theta\right]d\theta \\ &\quad + \int_{-\pi}^{0} g_-(\theta)\sin\left[\left(N+\tfrac{1}{2}\right)\theta\right]d\theta \qquad (12) \end{aligned}$$

by (5), where

$$g_{\pm}(\theta) = \frac{f(x+\theta) - f(x\pm)}{\sin\frac{1}{2}\theta}. \qquad (13)$$

The fourth step is to observe that the functions $\sin[(N+\frac{1}{2})\theta]$ $(N = 1, 2, 3, \ldots)$ form an orthogonal set on the interval $(-\pi, 0)$, as well as on the interval $(-0, \pi)$. Using Bessel's inequality as before, we deduce (see Exercise 8) that both of the integrals in (12) tend to zero as $N \to \infty$ provided that $\int_0^{\pi} |g_+(\theta)|^2\,d\theta$ and $\int_{-\pi}^{0} |g_-(\theta)|^2\,d\theta$ are finite.

That is the fifth step. The only possible reason for the divergence of these integrals would come from the vanishing of $\sin\frac{1}{2}\theta$ at $\theta = 0$. Now the *one-sided* limit of $g_+(\theta)$ is

$$\lim_{\theta\searrow 0} g_+(\theta) = \lim_{\theta\searrow 0} \frac{f(x+\theta) - f(x+)}{\theta} \cdot \frac{\theta}{\sin\left(\frac{1}{2}\theta\right)} = 2f'(x+) \qquad (14)$$

if x is a point where the one-sided derivative $f'(x+)$ exists. If $f'(x+)$ does not exist (e.g., f itself might have a jump at the point x), then f still is differentiable at nearby points. By the mean value theorem, $[f(x+\theta) - f(x+)]/\theta = f'(\theta^*)$ for some point θ^* between x and $x+\theta$. Since the derivative is bounded, it follows that $[f(x+\theta) - f(x)]/\theta$ is bounded as well for θ small and positive. So $g_+(\theta)$ is bounded and the integral $\int_0^{\pi} |g_+(\theta)|^2\,d\theta$ is finite. It works the same way for $g_-(\theta)$. □

PROOF OF UNIFORM CONVERGENCE

This is Theorem 5.4.2, for the case of classical Fourier series. We assume again that $f(x)$ and $f'(x)$ are continuous functions of period 2π. The idea of this proof is quite different from the preceding one. The main point is to show that the coefficients go to zero pretty fast. Let A_n and B_n be the Fourier coefficients of $f(x)$ and let A'_n and B'_n denote the Fourier coefficients of $f'(x)$. We integrate by parts to get

$$\begin{aligned} A_n &= \int_{-\pi}^{\pi} f(x)\cos nx \frac{dx}{\pi} \\ &= \frac{1}{n\pi} f(x)\sin nx\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f'(x)\sin nx \frac{dx}{n\pi}, \end{aligned}$$

so that

$$A_n = -\frac{1}{n}B'_n \qquad \text{for} \neq 0. \tag{15}$$

We have just used the periodicity of $f(x)$. Similarly,

$$B_n = \frac{1}{n}A'_n. \tag{16}$$

On the other hand, we know from Bessel's inequality [for the derivative $f'(x)$] that the infinite series

$$\sum_{n=1}^{\infty} \left(|A'_n|^2 + |B'_n|^2\right) < \infty.$$

Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} (|A_n \cos nx| + |B_n \sin nx|) &\le \sum_{n=1}^{\infty} (|A_n| + |B_n|) \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\left(|B'_n| + |A'_n|\right) \\ &\le \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2} \left[\sum_{n=1}^{\infty} 2\left(|A'_n|^2 + |B'_n|^2\right)\right]^{1/2} < \infty. \end{aligned}$$

Here we have used Schwarz's inequality (see Exercise 5). The result means that the Fourier series converges *absolutely*.

We already know (from Theorem 5.4.4∞) that the sum of the Fourier series is indeed $f(x)$. So, again denoting by $S_N(x)$ the partial sum (2), we can write

$$\begin{aligned} \max|f(x) - S_N(x)| &\le \max \sum_{n=N+1}^{\infty} |A_n \cos nx + B_n \sin nx| \\ &\le \sum_{n=N+1}^{\infty} (|A_n| + |B_n|) < \infty. \end{aligned} \tag{17}$$

The last sum is the tail of a convergent series of numbers so that it tends to zero as $N \to \infty$. Therefore, the Fourier series converges to $f(x)$ both absolutely and uniformly. □

This proof is also valid if $f(x)$ is continuous but $f'(x)$ is merely piecewise continuous. An example is $f(x) = |x|$.

THE GIBBS PHENOMENON

The Gibbs phenomenon is what happens to Fourier series at jump discontinuities. For a function with a jump, the partial sum $S_N(x)$ approximates the jump as in Figure 2 for a large value of N. Gibbs showed that $S_N(x)$ always differs from $f(x)$ near the jump by an "overshoot" of about 9 percent. The width of the overshoot goes to zero as $N \to \infty$ while the extra height remains at 9 percent (top and bottom). Thus

$$\lim_{N\to\infty} \max|S_N(x) - f(x)| \neq 0, \tag{18}$$

although $S_N(x) - f(x)$ does tend to zero for each x where $f(x)$ does not jump.

We now verify the Gibbs phenomenon for an example. Let's take the simplest odd function with a jump of unity; that is,

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } 0 < x < \pi \\ -\frac{1}{2} & \text{for } -\pi < x < 0, \end{cases}$$

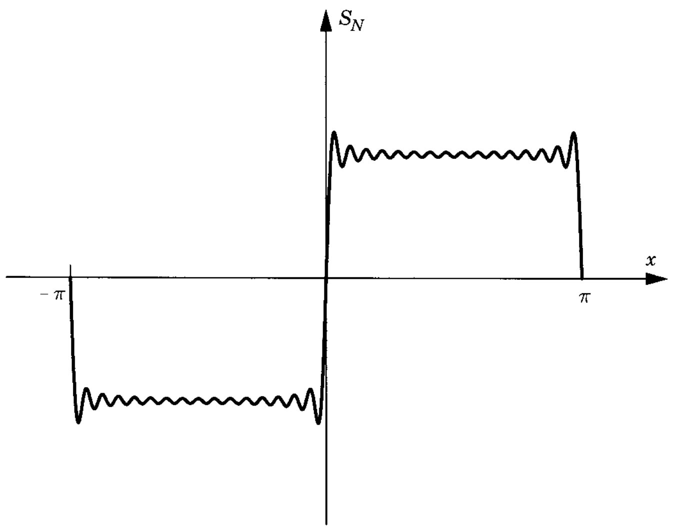

Figure 2

which has the Fourier series

$$\sum_{n \text{ odd}=1}^{\infty} \frac{2}{n\pi} \sin n\pi .$$

Figure 2 is a sketch of the partial sum $S_{16}(x)$. By (3) and (5), the partial sums are

$$\begin{aligned} S_N(x) &= \left(\int_0^{\pi} - \int_{-\pi}^{0} \right) K_N(x-y) \frac{dy}{4\pi} \\ &= \left(\int_0^{\pi} - \int_{-\pi}^{0} \right) \frac{\sin\left[\left(N+\frac{1}{2}\right)(x-y)\right]}{\sin\left[\frac{1}{2}(x-y)\right]} \frac{dy}{4\pi}. \end{aligned}$$

Let $M = N + \frac{1}{2}$. In the first integral let $\theta = M(x-y)$. In the second integral let $\theta = M(y-x)$. These changes of variables yield

$$\begin{aligned} S_N(x) &= \left(\int_{M(x-\pi)}^{Mx} - \int_{-M(x+\pi)}^{-Mx} \right) \frac{\sin\theta}{2M\sin(\theta/2M)} \frac{d\theta}{2\pi} \\ &= \left(\int_{-Mx}^{Mx} - \int_{-M\pi-Mx}^{-M\pi+Mx} \right) \frac{\sin\theta}{2M\sin(\theta/2M)} \frac{d\theta}{2\pi} \\ &= \left(\int_{-Mx}^{Mx} - \int_{M\pi-Mx}^{M\pi+Mx} \right) \frac{\sin\theta}{2M\sin(\theta/2M)} \frac{d\theta}{2\pi}, \end{aligned} \tag{19}$$

where we changed θ to $-\theta$ in the last step, the integrand being an even function.

We are interested in what happens near the jump, that is, where x is small. Remember that M is large. We will see that in (19) the first integral is the larger one because of the small denominator $\sin(\theta/2M)$. Where is the first integral in (19) maximized? Setting its derivative equal to zero, it is maximized where $\sin Mx = 0$. So we set $x = \pi/M$. Then (19) becomes

$$S_N\left(\frac{\pi}{M}\right) = \left(\int_{-\pi}^{\pi} - \int_{M\pi-\pi}^{M\pi+\pi} \right) \frac{\sin\theta}{2M\sin(\theta/2M)} \frac{d\theta}{2\pi}. \tag{20}$$

Inside the *second* integral in (20) the argument $\theta/2M$ is bounded both above and below, as follows:

$$\frac{\pi}{4} < \left[1 - \frac{1}{M}\right] \frac{\pi}{2} \le \frac{\theta}{2M} \le \left[1 + \frac{1}{M}\right] \frac{\pi}{2} < \frac{3\pi}{4}$$

for $M > 2$. Hence $\sin(\theta/2M) > 1/\sqrt{2}$, so that the second integral in (20) is less than

$$\int_{M\pi-\pi}^{M\pi+\pi} 1 \cdot \left[\frac{2M}{\sqrt{2}}\right]^{-1} \frac{d\theta}{2\pi} = \frac{1}{\sqrt{2}M},$$

which tends to zero as $M \to \infty$.

On the other hand, inside the *first* integral in (20) we have $|\theta| \leq \pi$ and

$$2M \sin\frac{\theta}{2M} \to \theta \qquad \text{uniformly in } -\pi \leq \theta \leq \pi \quad \text{as } M \to \infty.$$

Hence, taking the limit in (20) as $M \to \infty$, we get

$$S_N\left(\frac{\pi}{M}\right) \to \int_{-\pi}^{\pi} \frac{\sin\theta}{\theta}\frac{d\theta}{2\pi} \simeq 0.59. \tag{21}$$

This is Gibbs's 9 percent overshoot (of the unit jump value).

FOURIER SERIES SOLUTIONS

You could object, and you would be right, that we never showed that the Fourier series solutions actually solve the PDEs. Let's take a basic example to justify this final step. Consider the wave equation with Dirichlet boundary conditions and with initial conditions $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$ as in Section 4.1. The solution is supposed to be given by (4.1.9):

$$u(x, t) = \sum_n \left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}. \tag{22}$$

However, we know that term-by-term differentiation of a Fourier series is *not* always valid (see Example 3, Section 5.4), so we cannot simply verify by direct differentiation that (22) is a solution.

Instead, let ϕ_{ext} and ψ_{ext} denote the odd $2l$-periodic extensions of ϕ and ψ. Let us assume that ϕ and ψ are continuous with piecewise continuous derivatives. We know that the function

$$u(x, t) = \frac{1}{2}\left[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\, ds \tag{23}$$

solves the wave equation with $u(x, 0) = \phi_{\text{ext}}(x)$, $u_t(x, 0) = \psi_{\text{ext}}(x)$ for all $-\infty < x < \infty$. (Actually, it is a weak solution—see Section 12.1—but if we assume that ϕ_{ext} and ψ_{ext} are twice differentiable, it is an ordinary twice-differentiable solution.) Since ϕ_{ext} and ψ_{ext} agree with ϕ and ψ on the interval $(0, l)$, u satisfies the correct initial conditions on $(0, l)$. Since ϕ_{ext} and ψ_{ext} are odd, it follows that $u(x, t)$ is also odd, so that $u(0, t) = u(l, t) = 0$, which is the correct boundary condition.

By Theorem 5.4.4(i), the Fourier sine series of ϕ_{ext} and ψ_{ext}, given by (4.1.10) and (4.1.11), converge pointwise. Substituting these series into (23), we get

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} A_n\left(\sin\frac{n\pi(x+ct)}{l} + \sin\frac{n\pi(x-ct)}{l}\right)$$
$$+\frac{1}{2c}\sum_{n=1}^{\infty}\int_{x-ct}^{x+ct} B_n\frac{n\pi c}{l}\sin\frac{n\pi s}{l}\,ds. \qquad (24)$$

This series converges pointwise because term-by-term *integration* of a Fourier series is always valid, by Exercise 5.4.11. Now we use standard trigonometric identities and carry out the integrals explicitly. We get

$$u(x,t) = \sum_n \left(A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}\right). \qquad (25)$$

This is precisely (22).

EXERCISES

1. Sketch the graph of the Dirichlet kernel

$$K_N(\theta) = \frac{\sin\left(N+\frac{1}{2}\right)\theta}{\sin\frac{1}{2}\theta}$$

in case $N = 10$. Use a computer graphics program if you wish.

2. Prove the *Schwarz inequality* (for any pair of functions):

$$|(f,g)| \le \|f\|\cdot\|g\|.$$

(*Hint:* Consider the expression $\|f+tg\|^2$, where t is a scalar. This expression is a quadratic polynomial of t. Find the value of t where it is a minimum. Play around and the Schwarz inequality will pop out.)

3. Prove the inequality $l\int_0^l (f'(x))^2 dx \ge [f(l) - f(0)]^2$ for any real function $f(x)$ whose derivative $f'(x)$ is continuous. [*Hint:* Use Schwarz's inequality with the pair $f'(x)$ and 1.]

4. (a) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, $u(x, 0) = \phi(x)$, with the unusual boundary conditions

$$u_x(0,t) = u_x(l,t) = \frac{u(l,t) - u(0,t)}{l}.$$

Assume that there are no negative eigenvalues. (*Hint:* See Exercise 4.3.12.)

(b) Show that as $t \to \infty$,

$$\lim u(x,t) = A + Bx,$$

assuming that you can take limits term by term.

(c) Use Green's first identity and Exercise 3 to show that there are no negative eigenvalues.

(d) Find A and B. (*Hint:* $A + Bx$ is the beginning of the series. Take the inner product of the series for $\phi(x)$ with each of the functions 1 and x. Make use of the orthogonality.)

5. Prove the *Schwarz inequality for infinite series:*
$$\sum a_n b_n \le \left(\sum a_n^2\right)^{1/2} \left(\sum b_n^2\right)^{1/2}.$$
(*Hint:* See the hint in Exercise 2. Prove it first for finite series (ordinary sums) and then pass to the limit.)
6. Consider the diffusion equation on $[0, l]$ with Dirichlet boundary conditions and any continuous function as initial condition. Show from the series expansion that the solution is infinitely differentiable for $t > 0$. (*Hint:* Use the general theorem at the end of Section A.2 on the differentiability of series, together with the fact that the exponentials are very small for large n. See Section 3.5 for an analogous situation.)
7. Let $\int_{-\pi}^{\pi} [|f(x)|^2 + |g(x)|^2]\,dx$ be finite, where $g(x) = f(x)/(e^{ix} - 1)$. Let c_n be the coefficients of the full complex Fourier series of $f(x)$. Show that $\Sigma_{n=-N}^{N} c_n \to 0$ as $N \to \infty$.
8. Prove that both integrals in (12) tend to zero.
9. Fill in the missing steps in the proof of uniform convergence.
10. Prove the theorem on uniform convergence for the case of the Fourier sine series and for the Fourier cosine series.
11. Prove that the classical full Fourier series of $f(x)$ converges *uniformly* to $f(x)$ if merely $f(x)$ is continuous of period 2π and its derivative $f'(x)$ is *piecewise* continuous. (*Hint*: Modify the discussion of uniform convergence in this section.)
12. Show that if $f(x)$ is a C^1 function in $[-\pi, \pi]$ that satisfies the periodic BC and if $\int_{-\pi}^{\pi} f(x)\,dx = 0$, then $\int_{-\pi}^{\pi} |f|^2\,dx \le \int_{-\pi}^{\pi} |f'|^2\,dx$. (*Hint:* Use Parseval's equality.)
13. A very slick proof of the pointwise convergence of Fourier series, due to P. Chernoff (*American Mathematical Monthly*, May 1980), goes as follows.
    (a) Let $f(x)$ be a C^1 function of period 2π. First show that we may as well assume that $f(0) = 0$ and we need only show that the Fourier series converges to zero at $x = 0$.
    (b) Let $g(x) = f(x)/(e^{ix} - 1)$. Show that $g(x)$ is a continuous function.
    (c) Let C_n be the (complex) Fourier coefficients of $f(x)$ and D_n the coefficients of $g(x)$. Show that $D_n \to 0$.
    (d) Show that $C_n = D_{n-1} - D_n$ so that the series ΣC_n is telescoping.
    (e) Deduce that the Fourier series of $f(x)$ at $x = 0$ converges to zero.
14. Prove the validity of the Fourier series solution of the diffusion equation on $(0, l)$ with $u_x(x, 0) = u_x(x, l) = 0$, $u(x, 0) = \phi(x)$, where $\phi(x)$ is continuous with a piecewise continuous derivative. That is, prove that the series truly converges to the solution.
15. Carry out the step going from (24) to (25).

5.6 INHOMOGENEOUS BOUNDARY CONDITIONS

In this section we consider problems with sources given at the boundary. We shall see that naive use of the separation of variables technique will not work.

Let's begin with the *diffusion equation with sources at both endpoints*.

$$\begin{gathered} u_t = ku_{xx} \qquad 0 < x < l, \quad t > 0 \\ \boldsymbol{u(0,t) = h(t) \qquad u(l,t) = j(t)} \\ u(x, 0) \equiv 0. \end{gathered} \tag{1}$$

A separated solution $u = X(x)T(t)$ just will not fit the boundary conditions. So we try a slightly different approach.

EXPANSION METHOD

We already know that for the corresponding homogeneous problem the correct expansion is the Fourier sine series. For each t, we certainly can expand

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l}} \tag{2}$$

for *some* coefficients $u_n(t)$, because the completeness theorems guarantee that any function in $(0, l)$ can be so expanded. The coefficients are necessarily given by

$$u_n(t) = \frac{2}{l} \int_0^l u(x,t) \sin \frac{n\pi x}{l} \, dx. \tag{3}$$

You may object that each term in the series vanishes at both endpoints and thereby violates the boundary conditions. The answer is that we simply do not insist that the series converge at the endpoints but only inside the interval. In fact, we are exactly in the situation of Theorems 3 and 4 but not of Theorem 2 of Section 5.4.

Now differentiating the series (2) term by term, we get

$$0 = u_t - ku_{xx} = \sum \left[\frac{du_n}{dt} + ku_n(t) \left(\frac{n\pi}{l}\right)^2 \right] \sin \frac{n\pi x}{l}.$$

So the PDE seems to require that $du_n/dt + k\lambda_n u_n = 0$, so that $u_n(t) = A_n e^{k\lambda_n t}$. There is no way for this to fit the boundary conditions. Our method fails! What's the moral? It is that you can't differentiate term by term. See Example 3 in Section 5.4 for the dangers of differentiation.

Let's start over again but avoid direct differentiation of the Fourier series. The expansion (2) with the coefficients (3) must be valid, by the completeness theorem 5.4.3, say, provided that $u(x, t)$ is a continuous function. Clearly, the initial condition requires that $u_n(0) = 0$. If the derivatives of $u(x, t)$ are also continuous, let's expand them, too. Thus

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} v_n(t) \sin \frac{n\pi x}{l} \tag{4}$$

with

$$v_n(t) = \frac{2}{l} \int_0^l \frac{\partial u}{\partial t} \sin \frac{n\pi x}{l}\, dx = \frac{du_n}{dt}. \tag{5}$$

The last equality is valid since we can differentiate under an integral sign if the new integrand is continuous (see Section A.3). We also expand

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=1}^{\infty} w_n(t) \sin \frac{n\pi x}{l} \tag{6}$$

with the coefficients

$$w_n(t) = \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{l}\, dx. \tag{7}$$

By Green's second identity (5.3.3) the last expression equals

$$\frac{-2}{l} \int_0^l \left(\frac{n\pi}{l}\right)^2 u(x,t) \sin \frac{n\pi x}{l}\, dx + \frac{2}{l} \left(u_x \sin \frac{n\pi x}{l} - \frac{n\pi}{l} u \cos \frac{n\pi x}{l} \right) \Bigg|_0^l.$$

Here come the boundary conditions. The sine factor vanishes at both ends. The last term will involve the boundary conditions. Thus

$$w_n(t) = -\lambda_n u_n(t) - 2n\pi l^{-2}(-1)^n j(t) + 2n\pi l^{-2} h(t), \tag{8}$$

where $\lambda_n = (n\pi/l)^2$. Now by (5) and (7) the PDE requires

$$v_n(t) - k w_n(t) = \frac{2}{l} \int_0^l (u_t - k u_{xx}) \sin \frac{n\pi x}{l}\, dx = \int_0^l 0 = 0.$$

So from (5) and (8) we deduce that $u_n(t)$ satisfies

$$\frac{du_n}{dt} = k\{-\lambda_n u_n(t) - 2n\pi l^{-2}[(-1)^n j(t) - h(t)]\}. \tag{9}$$

This is just an ordinary differential equation, to be solved together with the initial condition $u_n(0) = 0$ from (1). The solution of (9) is

$$\boxed{u_n(t) = Ce^{-\lambda_n kt} - 2n\pi l^{-2} k \int_0^t e^{-\lambda_n k(t-s)}[(-1)^n j(s) - h(s)]\, ds.} \tag{10}$$

As a second case, let's solve the *inhomogeneous wave problem*

$$\begin{gathered} u_{tt} - c^2 u_{xx} = f(x,t) \\ \boldsymbol{u(0,t) = h(t) \qquad u(l,t) = k(t)} \\ u(x,0) = \phi(x) \qquad u_t(x,0) = \psi(x). \end{gathered} \tag{11}$$

Again we expand everything in the eigenfunctions of the corresponding homogeneous problem:

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin \frac{n\pi x}{l},$$

$u_{tt}(x,t)$ with coefficients $v_n(t)$, $u_{xx}(x,t)$ with coefficients $w_n(t)$, $f(x,t)$ with coefficients $f_n(t)$, $\phi(x)$ with coefficients ϕ_n, and $\psi(x)$ with coefficients ψ_n. Then

$$v_n(t) = \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial t^2} \sin \frac{n\pi x}{l}\, dx = \frac{d^2 u_n}{dt^2}$$

and, just as before,

$$\begin{aligned} w_n(t) &= \frac{2}{l} \int_0^l \frac{\partial^2 u}{\partial x^2} \sin \frac{n\pi x}{l}\, dx \\ &= -\lambda_n u_n(t) + 2n\pi l^{-2}[h(t) - (-1)^n k(t)]. \end{aligned}$$

From the PDE we also have

$$v_n(t) - c^2 w_n(t) = \frac{2}{l} \int_0^l (u_{tt} - c^2 u_{xx}) \sin \frac{n\pi x}{l}\, dx = f_n(t).$$

Therefore,

$$\frac{d^2 u_n}{dt^2} + c^2 \lambda_n u_n(t) = -2n\pi l^{-2} \left[(-1)^n k(t) - h(t)\right] + f_n(t) \tag{12}$$

with the initial conditions

$$u_n(0) = \phi_n \qquad u_n'(0) = \psi_n.$$

The solution can be written explicitly (see Exercise 11).

METHOD OF SHIFTING THE DATA

By subtraction, the data can be shifted from the boundary to another spot in the problem. *The boundary conditions can be made homogeneous by subtracting any known function that satisfies them*. Thus for the problem (11) treated above, the function

$$\mathcal{U}(x,t) = \left(1 - \frac{x}{l}\right) h(t) + \frac{x}{l} k(t)$$

obviously satisfies the BCs. If we let

$$v(x, t) = u(x, t) - \mathcal{U}(x, t),$$

then $v(x, t)$ satisfies the same problem but with zero boundary data, with initial data $\phi(x) - \mathcal{U}(x, 0)$ and $\psi(x) - \mathcal{U}_t(x, 0)$, and with right-hand side f replaced by $f - \mathcal{U}_{tt}$.

The boundary condition and the differential equation can simultaneously be made homogeneous by subtracting any known function that satisfies them. One case when this can surely be accomplished is the case of "stationary data" when h, k, and $f(x)$ all are independent of time. Then it is easy to find a solution of

$$-c^2\mathcal{U}_{xx} = f(x) \qquad \mathcal{U}(0) = h \qquad \mathcal{U}(l) = k.$$

Then $v(x, t) = u(x, t) - \mathcal{U}(x)$ solves the problem with zero boundary data, zero right-hand side, and initial data $\phi(x) - \mathcal{U}(x)$ and $\psi(x)$.

For another example, take problem (11) for a simple periodic case:

$$f(x, t) = F(x)\cos\omega t \qquad h(t) = H\cos\omega t \qquad k(t) = K\cos\omega t,$$

that is, with the same time behavior in all the data. We wish to subtract a solution of

$$\mathcal{U}_{tt} - c^2\mathcal{U}_{xx} = F(x)\cos\omega t$$
$$\mathcal{U}(0, t) = H\cos\omega t \qquad \mathcal{U}(l, t) = K\cos\omega t.$$

A good guess is that $\mathcal{U}$ should have the form $\mathcal{U}(x, t) = \mathcal{U}_0(x)\cos\omega t$. This will happen if $\mathcal{U}_0(x)$ satisfies

$$-\omega^2\mathcal{U}_0 - c^2\mathcal{U}_0'' = F(x) \qquad \mathcal{U}_0(0) = H \qquad \mathcal{U}_0(l) = K. \qquad \square$$

There is also the method of Laplace transforms, which can be found in Section 12.5.

EXERCISES

1. (a) Solve as a series the equation $u_t = u_{xx}$ in (0, 1) with $u_x(0, t) = 0$, $u(1, t) = 1$, and $u(x, 0) = x^2$. Compute the first two coefficients explicitly.
   (b) What is the equilibrium state (the term that does not tend to zero)?
2. For problem (1), complete the calculation of the series in case $j(t) = 0$ and $h(t) = e^t$.
3. Repeat problem (1) for the case of Neumann BCs.
4. Solve $u_{tt} = c^2u_{xx} + k$ for $0 < x < l$, with the boundary conditions $u(0, t) = 0$, $u_x(l, t) = 0$ and the initial conditions $u(x, 0) = 0$, $u_t(x, 0) = V$. Here k and V are constants.
5. Solve $u_{tt} = c^2u_{xx} + e^t\sin 5x$ for $0 < x < \pi$, with $u(0, t) = u(\pi, t) = 0$ and the initial conditions $u(x, 0) = 0$, $u_t(x, 0) = \sin 3x$.

6. Solve $u_{tt} = c^2 u_{xx} + g(x)\sin\omega t$ for $0 < x < l$, with $u = 0$ at both ends and $u = u_t = 0$ when $t = 0$. For which values of ω can resonance occur? (Resonance means growth in time.)
7. Repeat Exercise 6 for the damped wave equation $u_{tt} = c^2 u_{xx} - ru_t + g(x)\sin\omega t$, where r is a positive constant.
8. Solve $u_t = ku_{xx}$ in $(0, l)$, with $u(0, t) = 0$, $u(l, t) = At$, $u(x, 0) = 0$, where A is a constant.
9. Use the method of subtraction to solve $u_{tt} = 9u_{xx}$ for $0 \le x \le 1 = l$, with $u(0, t) = h$, $u(1, t) = k$, where h and k are given constants, and $u(x, 0) = 0$, $u_t(x, 0) = 0$.
10. Find the temperature of a metal rod that is in the shape of a solid circular cone with cross-sectional area $A(x) = b(1 - x/l)^2$ for $0 \le x \le l$, where b is a constant. Assume that the rod is made of a uniform material, is insulated on its sides, is maintained at zero temperature on its flat end ($x = 0$), and has an unspecified initial temperature distribution $\phi(x)$. Assume that the temperature is independent of y and z. [*Hint:* Derive the PDE $(1 - x/l)^2 u_t = k\{(1 - x/l)^2 u_x\}_x$. Separate variables $u = T(t)X(x)$ and then substitute $v(x) = (1 - x/l)X(x)$.]
11. Write out the solution of problem (11) explicitly, starting from the discussion in Section 5.6.
12. Carry out the solution of (11) in the case that

$$f(x, t) = F(x)\cos\omega t \qquad h(t) = H\cos\omega t \qquad k(t) = K\cos\omega t.$$

13. If friction is present, the wave equation takes the form

$$u_{tt} - c^2 u_{xx} = -ru_t,$$

where the resistance $r > 0$ is a constant. Consider a periodic source at one end: $u(0, t) = 0$, $u(l, t) = Ae^{i\omega t}$.

(a) Show that the PDE and the BC are satisfied by

$$\mathcal{U}(x, t) = Ae^{i\omega t}\frac{\sin\beta x}{\sin\beta l}, \qquad \text{where } \beta^2 c^2 = \omega^2 - ir\omega.$$

(b) No matter what the IC, $u(x, 0)$ and $u_t(x, 0)$, are, show that $\mathcal{U}(x, t)$ is the asymptotic form of the solution $u(x, t)$ as $t \to \infty$.

(c) Show that you can get resonance as $r \to 0$ if $\omega = m\pi c/l$ for some integer m.

(d) Show that friction can prevent resonance from occurring.