4

BOUNDARY PROBLEMS

In this chapter we finally come to the physically realistic case of a finite interval $0 < x < l$. The methods we introduce will frequently be used in the rest of this book.

4.1 SEPARATION OF VARIABLES, THE DIRICHLET CONDITION

We first consider the homogeneous Dirichlet conditions for the wave equation:

$$\boxed{\begin{aligned} u_{tt} &= c^2 u_{xx} \quad \text{for } 0 < x < l && (1)\\ u(0,t) &= 0 = u(l,t) && (2)\end{aligned}}$$

with some initial conditions

$$\boxed{u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x).} \tag{3}$$

The method we shall use consists of building up the general solution as a linear combination of special ones that are easy to find. (Once before, in Section 2.4, we followed this program, but with different building blocks.)

A *separated solution* is a solution of (1) and (2) of the form

$$u(x,t) = X(x)T(t). \tag{4}$$

(It is important to distinguish between the independent variable written as a lowercase letter and the function written as a capital letter.) Our first goal is to look for as many separated solutions as possible.

Plugging the form (4) into the wave equation (1), we get

$$X(x)T''(t) = c^2 X''(x)T(t)$$

or, dividing by $-c^2 XT$,

$$-\frac{T''}{c^2 T} = -\frac{X''}{X} = \lambda.$$

This defines a quantity λ, *which must be a constant*. (*Proof:* $\partial\lambda/\partial x = 0$ and $\partial\lambda/\partial t = 0$, so λ is a constant. Alternatively, we can argue that λ doesn't depend on x because of the first expression and doesn't depend on t because of the second expression, so that it doesn't depend on any variable.) We will show at the end of this section that $\lambda > 0$. (This is the reason for introducing the minus signs the way we did.)

So let $\lambda = \beta^2$, where $\beta > 0$. Then the equations above are a pair of *separate* (!) ordinary differential equations for $X(x)$ and $T(t)$:

$$X'' + \beta^2 X = 0 \quad \text{and} \quad T'' + c^2\beta^2 T = 0. \tag{5}$$

These ODEs are easy to solve. The solutions have the form

$$X(x) = C\cos\beta x + D\sin\beta x \tag{6}$$

$$T(t) = A\cos\beta ct + B\sin\beta ct, \tag{7}$$

where A, B, C, and D are constants.

The second step is to impose the boundary conditions (2) on the separated solution. They simply require that $X(0) = 0 = X(l)$. Thus

$$0 = X(0) = C \quad \text{and} \quad 0 = X(l) = D\sin\beta l.$$

Surely we are not interested in the obvious solution $C = D = 0$. So we must have $\beta l = n\pi$, a root of the sine function. That is,

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l} \quad (n = 1, 2, 3, \ldots) \tag{8}$$

are distinct solutions. Each sine function may be multiplied by an arbitrary constant.

Therefore, there are an *infinite* (!) number of separated solutions of (1) and (2), one for each n. They are

$$u_n(x,t) = \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

$(n = 1, 2, 3, \ldots)$, where A_n and B_n are arbitrary constants. The sum of solutions is again a solution, so *any finite sum*

$$\boxed{u(x,t) = \sum_n \left(A_n\cos\frac{n\pi ct}{l} + B_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}} \tag{9}$$

is also a solution of (1) and (2).

Formula (9) solves (3) as well as (1) and (2), provided that

$$\phi(x) = \sum_n A_n \sin \frac{n\pi x}{l} \tag{10}$$

and

$$\psi(x) = \sum_n \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}. \tag{11}$$

Thus for any initial data of this form, the problem (1), (2), and (3) has a simple explicit solution.

But such data (10) and (11) clearly are very special. So let's try (following Fourier in 1827) to take *infinite sums*. Then we ask what kind of data pairs $\phi(x)$, $\psi(x)$ can be expanded as in (10), (11) for some choice of coefficients A_n, B_n? This question was the source of great disputes for half a century around 1800, but the final result of the disputes was very simple: *Practically any* (!) *function* $\phi(x)$ *on the interval* $(0, l)$ *can be expanded in an infinite series* (10). We will show this in Chapter 5. It will have to involve technical questions of convergence and differentiability of infinite series like (9). The series in (10) is called a *Fourier sine series* on $(0, l)$. But for the time being let's not worry about these mathematical points. Let's just forge ahead to see what their implications are.

First of all, (11) is the same kind of series for $\psi(x)$ as (10) is for $\phi(x)$. What we've shown is simply that *if* (10), (11) *are true, then the infinite series* (9) *ought to be the solution of the whole problem* (1), (2), (3).

A sketch of the first few functions $\sin(\pi x/l)$, $\sin(2\pi x/l)$, ... is shown in Figure 1. The functions $\cos(n\pi ct/l)$ and $\sin(n\pi ct/l)$ which describe the

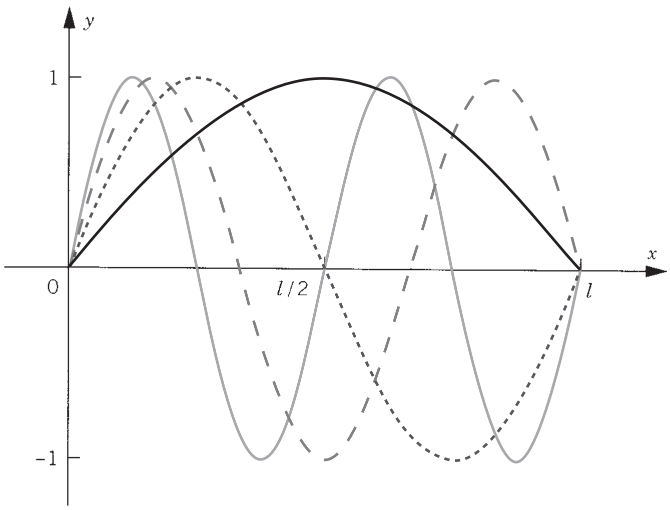

Figure 1

behavior in *time* have a similar form. The coefficients of t inside the sines and cosines, namely $n\pi c/l$, are called the *frequencies*. (In some texts, the frequency is defined as $nc/2l$.)

If we return to the violin string that originally led us to the problem (1), (2), (3), we find that the frequencies are

$$\frac{n\pi\sqrt{T}}{l\sqrt{\rho}} \quad \text{for } n = 1, 2, 3, \ldots \tag{12}$$

The "fundamental" note of the string is the smallest of these, $\pi\sqrt{T}/(l\sqrt{\rho})$. The "overtones" are *exactly* the double, the triple, and so on, of the fundamental! The discovery by Euler in 1749 that the musical notes have such a simple mathematical description created a sensation. It took over half a century to resolve the ensuing controversy over the relationship between the infinite series (9) and d'Alembert's solution in Section 2.1. □

The analogous problem for *diffusion* is

$$\boxed{\begin{aligned} &\text{DE:} \quad u_t = ku_{xx} \quad (0 < x < l, 0 < t < \infty) \\ &\text{BC:} \quad u(0,t) = u(l,t) = 0 \\ &\text{IC:} \quad u(x,0) = \phi(x). \end{aligned}} \qquad \begin{aligned} (13) \\ (14) \\ (15) \end{aligned}$$

To solve it, we separate the variables $u = T(t)X(x)$ as before. This time we get

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = \text{constant}.$$

Therefore, $T(t)$ satisfies the equation $T' = -\lambda kT$, whose solution is $T(t) = Ae^{-\lambda kt}$. Furthermore,

$$-X'' = \lambda X \quad \text{in } 0 < x < l \quad \text{with} \quad X(0) = X(l) = 0. \tag{16}$$

This is *precisely the same problem for* $X(x)$ *as before* and so has the same solutions. Because of the form of $T(t)$,

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin\frac{n\pi x}{l}} \tag{17}$$

is the solution of (13)–(15) provided that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}. \tag{18}$$

Once again, our solution is expressible for each t as a Fourier sine series in x provided that the initial data are.

For example, consider the diffusion of a substance in a tube of length l. Each end of the tube opens up into a very large empty vessel. So the concentration $u(x, t)$ at each end is essentially zero. Given an initial concentration $\phi(x)$ in the tube, the concentration at all later times is given by formula (17). Notice that as $t \to \infty$, each term in (17) goes to zero. Thus the substance gradually empties out into the two vessels and less and less remains in the tube. □

The numbers $\lambda_n = (\boldsymbol{n\pi/l})^2$ are called *eigenvalues* and the functions $X_n(x) = \mathbf{sin}(\boldsymbol{n\pi x/l})$ are called *eigenfunctions*. The reason for this terminology is as follows. They satisfy the conditions

$$-\frac{d^2}{dx^2}X = \lambda X, \quad X(0) = X(l) = 0. \tag{19}$$

This is an ODE with conditions at two points. Let A denote the operator $-d^2/dx^2$, which acts on the functions that satisfy the Dirichlet boundary conditions. The differential equation has the form $AX = \lambda X$. An eigenfunction is a solution $X \not\equiv 0$ of this equation and an eigenvalue is a number λ for which there exists a solution $X \not\equiv 0$.

This situation is analogous to the more familiar case of an $N \times N$ matrix A. A vector X that satisfies $AX = \lambda X$ with $X \not\equiv 0$ is called an eigenvector and λ is called an eigenvalue. For an $N \times N$ matrix there are at most N eigenvalues. But for the differential operator that we are interested in, there are an *infinite number of eigenvalues* π^2/l^2, $4\pi^2/l^2$, $9\pi^2/l^2$, ... Thus you might say that we are dealing with *infinite-dimensional linear algebra!*

In physics and engineering the eigenfunctions are called *normal modes* because they are the natural shapes of solutions that persist for all time.

Why are all the eigenvalues of this problem positive? We assumed this in the discussion above, but now let's *prove* it. First, could $\lambda = 0$ be an eigenvalue? This would mean that $X'' = 0$, so that $X(x) = C + Dx$. But $X(0) = X(l) = 0$ implies that $C = D = 0$, so that $X(x) \equiv 0$. Therefore, zero is *not* an eigenvalue.

Next, could there be *negative* eigenvalues? If $\lambda < 0$, let's write it as $\lambda = -\gamma^2$. Then $X'' = \gamma^2 X$, so that $X(x) = C \cosh \gamma x + D \sinh \gamma x$. Then $0 = X(0) = C$ and $0 = X(l) = D \sinh \gamma l$. Hence $D = 0$ since $\sinh \gamma l \neq 0$.

Finally, let λ be any *complex* number. Let γ be either one of the two square roots of $-\lambda$; the other one is $-\gamma$. Then

$$X(x) = Ce^{\gamma x} + De^{-\gamma x},$$

where we are using the complex exponential function (see Section 5.2). The boundary conditions yield $0 = X(0) = C + D$ and $0 = Ce^{\gamma l} + De^{-\gamma l}$. Therefore $e^{2\gamma l} = 1$. By a well-known property of the complex exponential function, this implies that $\mathrm{Re}(\gamma) = 0$ and $2l\ \mathrm{Im}(\gamma) = 2\pi n$ for some integer n. Hence $\gamma = n\pi i/l$ and $\lambda = -\gamma^2 = n^2\pi^2/l^2$, which is real and positive. Thus the only eigenvalues λ of our problem (16) are positive numbers; in fact, they are $(\pi/l)^2$, $(2\pi/l)^2$, ...

EXERCISES

1. (a) Use the Fourier expansion to explain why the note produced by a violin string rises sharply by one octave when the string is clamped exactly at its midpoint.
   (b) Explain why the note rises when the string is tightened.
2. Consider a metal rod ($0 < x < l$), insulated along its sides but not at its ends, which is initially at temperature $= 1$. Suddenly both ends are plunged into a bath of temperature $= 0$. Write the differential equation, boundary conditions, and initial condition. Write the formula for the temperature $u(x, t)$ at later times. In this problem, *assume* the infinite series expansion
$$1 = \frac{4}{\pi}\left(\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right)$$
3. A quantum-mechanical particle on the line with an infinite potential outside the interval $(0, l)$ ("particle in a box") is given by Schrödinger's equation $u_t = iu_{xx}$ on $(0, l)$ with Dirichlet conditions at the ends. Separate the variables and use (8) to find its representation as a series.
4. Consider waves in a resistant medium that satisfy the problem
$$u_{tt} = c^2u_{xx} - ru_t \quad \text{for } 0 < x < l$$
$$u = 0 \quad \text{at both ends}$$
$$u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x),$$
where r is a constant, $0 < r < 2\pi c/l$. Write down the series expansion of the solution.
5. Do the same for $2\pi c/l < r < 4\pi c/l$.
6. Separate the variables for the equation $tu_t = u_{xx} + 2u$ with the boundary conditions $u(0, t) = u(\pi, t) = 0$. Show that there are an infinite number of solutions that satisfy the initial condition $u(x, 0) = 0$. So uniqueness is false for this equation!

4.2 THE NEUMANN CONDITION

The same method works for both the Neumann and Robin boundary conditions (BCs). In the former case, (4.1.2) is replaced by $u_x(0, t) = u_x(l, t) = 0$. Then the eigenfunctions are the solutions $X(x)$ of

$$\boxed{-X'' = \lambda X, \quad X'(0) = X'(l) = 0,} \tag{1}$$

other than the trivial solution $X(x) \equiv 0$.

As before, let's first search for the positive eigenvalues $\lambda = \beta^2 > 0$. As in (4.1.6), $X(x) = C\cos\beta x + D\sin\beta x$, so that

$$X'(x) = -C\beta\sin\beta x + D\beta\cos\beta x.$$

The boundary conditions (1) mean first that $0 = X'(0) = D\beta$, so that $D = 0$, and second that

$$0 = X'(l) = -C\beta \sin \beta l.$$

Since we don't want $C = 0$, we must have $\sin \beta l = 0$. Thus $\beta = \pi/l,\ 2\pi/l,\ 3\pi/l, \ldots$. Therefore, we have the

$$\boxed{\begin{array}{ll} \text{Eigenvalues:} & \left(\dfrac{\pi}{l}\right)^2, \left(\dfrac{2\pi}{l}\right)^2, \cdots \qquad (2) \\ \text{Eigenfunctions:} & X_n(x) = \cos \dfrac{n\pi x}{l} \quad (n = 1, 2, \ldots) \qquad (3) \end{array}}$$

Next let's check whether zero is an eigenvalue. Set $\lambda = 0$ in the ODE (1). Then $X'' = 0$, so that $X(x) = C + Dx$ and $X'(x) \equiv D$. The Neumann boundary conditions are both satisfied if $D = 0$. C can be any number. Therefore, $\lambda = 0$ *is an eigenvalue*, and any constant function is its eigenfunction.

If $\lambda < 0$ or if λ is complex (nonreal), it can be shown directly, as in the Dirichlet case, that there is no eigenfunction. (Another proof will be given in Section 5.3.) Therefore, the list of all the eigenvalues is

$$\boxed{\lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad \text{for } n = 0, 1, 2, 3, \ldots.} \tag{4}$$

Note that $n = 0$ *is included among them!*

So, for instance, the *diffusion* equation with the Neumann BCs has the solution

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos \frac{n\pi x}{l}.} \tag{5}$$

This solution requires the initial data to have the "Fourier cosine expansion"

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}. \tag{6}$$

All the coefficients $A_0, A_1, A_2, \ldots$ are just constants. The first term in (5) and (6), which comes from the eigenvalue $\lambda = 0$, is written separately in the form $\frac{1}{2}A_0$ just for later convenience. (The reader is asked to bear with this ridiculous factor $\frac{1}{2}$ until Section 5.1 when its convenience will become apparent.)

What is the behavior of $u(x, t)$ as $t \to +\infty$? Since all but the first term in (5) contains an exponentially decaying factor, the solution decays quite fast to the first term $\frac{1}{2}A_0$, which is just a constant. Since these boundary conditions correspond to insulation at both ends, this agrees perfectly with our intuition of Section 2.5 that the solution "spreads out." This is the eventual behavior if we wait long enough. (To actually *prove* that the limit as $t \to \infty$ is given term by term in (5) requires the use of one of the convergence theorems in Section A.2. We omit this verification here.)

Consider now the *wave* equation with the Neumann BCs. The eigenvalue $\lambda = 0$ then leads to $X(x) =$ constant and to the differential equation $T''(t) = \lambda c^2 T(t) = 0$, which has the solution $T(t) = A + Bt$. Therefore, the wave equation with Neumann BCs has the solutions

$$\boxed{\begin{aligned} u(x,t) = \frac{1}{2}A_0 &+ \frac{1}{2}B_0 t \\ &+ \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi ct}{l}\right)\cos\frac{n\pi x}{l}. \end{aligned}} \tag{7}$$

(Again, the factor $\frac{1}{2}$ will be justified later.) Then the initial data must satisfy

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} \tag{8}$$

and

$$\psi(x) = \frac{1}{2}B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos\frac{n\pi x}{l}. \tag{9}$$

Equation (9) comes from first differentiating (7) with respect to t and then setting $t = 0$. □

A "mixed" boundary condition would be Dirichlet at one end and Neumann at the other. For instance, in case the BCs are $u(0,t) = u_x(l,t) = 0$, the eigenvalue problem is

$$\boxed{-X'' = \lambda X \qquad X(0) = X'(l) = 0.} \tag{10}$$

The eigenvalues then turn out to be $(n+\frac{1}{2})^2\pi^2/l^2$ and the eigenfunctions $\sin[(n+\frac{1}{2})\pi x/l]$ for $n = 0, 1, 2, \ldots$ (see Exercises 1 and 2). For a discussion of boundary conditions in the context of musical instruments, see [HJ].

For another example, consider the *Schrödinger* equation $u_t = iu_{xx}$ in $(0, l)$ with the Neumann BCs $u_x(0,t) = u_x(l,t) = 0$ and initial condition $u(x,0) = \phi(x)$. Separation of variables leads to the equation

$$\frac{T'}{iT} = \frac{X''}{X} = -\lambda = \text{constant},$$

so that $T(t) = e^{-i\lambda t}$ and $X(x)$ satisfies exactly the same problem (1) as before. Therefore, the solution is

$$\boxed{u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-i(n\pi/l)^2 t} \cos\frac{n\pi x}{l}.}$$

The initial condition requires the cosine expansion (6).

EXERCISES

1. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.
2. Consider the equation $u_{tt} = c^2 u_{xx}$ for $0 < x < l$, with the boundary conditions $u_x(0, t) = 0$, $u(l, \text{t}) = 0$ (Neumann at the left, Dirichlet at the right).
   (a) Show that the eigenfunctions are $\cos[(n+\frac{1}{2})\pi x/l]$.
   (b) Write the series expansion for a solution $u(x,t)$.
3. Solve the Schrödinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < l$ with the boundary conditions $u_x(0,t) = 0$, $u(l,t) = 0$.
4. Consider diffusion inside an enclosed circular tube. Let its length (circumference) be $2l$. Let x denote the arc length parameter where $-l \le x \le l$. Then the concentration of the diffusing substance satisfies

$$u_t = ku_{xx} \quad \text{for } -l \le x \le l$$

$$u(-l,t) = u(l,t) \quad \text{and} \quad u_x(-l,t) = u_x(l,t).$$

   These are called *periodic boundary conditions*.
   (a) Show that the eigenvalues are $\lambda = (n\pi/l)^2$ for $n = 0, 1, 2, 3, \ldots$.
   (b) Show that the concentration is

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{l} + B_n \sin\frac{n\pi x}{l}\right)e^{-n^2\pi^2 kt/l^2}.$$

4.3 THE ROBIN CONDITION

We continue the method of separation of variables for the case of the Robin condition. The Robin condition means that we are solving $-X'' = \lambda X$ with the boundary conditions

$$\boxed{\begin{aligned} X' - a_0 X &= 0 \qquad \text{at } x = 0 \\ X' + a_l X &= 0 \qquad \text{at } x = l. \end{aligned}} \qquad \begin{aligned}(1)\\(2)\end{aligned}$$

The two constants a_0 and a_l should be considered as given.

The physical reason they are written with opposite signs is that they correspond to *radiation* of energy if a_0 and a_l are positive, *absorption* of energy if a_0 and a_l are negative, and *insulation* if $a_0 = a_l = 0$. This is the interpretation for a heat problem: See the discussion in Section 1.4 or Exercise 2.3.8. For the case of the vibrating string, the interpretation is that the string shares its energy with the endpoints if a_0 and a_l are positive, whereas the string gains some energy from the endpoints if a_0 and a_l are negative: See Exercise 11.

The mathematical reason for writing the constants in this way is that the unit *outward* normal **n** for the interval $0 \le x \le l$ points to the *left* at $x = 0$ ($\mathbf{n} = -1$) and to the *right* at $x = l$ ($\mathbf{n} = +1$). Therefore, we expect that the nature of the eigenfunctions might depend on the signs of the two constants in opposite ways.

POSITIVE EIGENVALUES

Our task now is to solve the ODE $-X'' = \lambda X$ with the boundary conditions (1), (2). First let's look for the *positive eigenvalues*

$$\lambda = \beta^2 > 0.$$

As usual, the solution of the ODE is

$$X(x) = C\ \cos \beta x + D\ \sin \beta x \tag{3}$$

so that

$$X'(x) \pm aX(x) = (\beta D \pm aC)\ \cos \beta x + (-\beta C \pm aD)\ \sin \beta x.$$

At the left end $x = 0$ we require that

$$0 = X'(0) - a_0 X(0) = \beta D - a_0 C. \tag{4}$$

So we can solve for D in terms of C. At the right end $x = l$ we require that

$$0 = (\beta D + a_l C)\ \cos \beta l + (-\beta C + a_l D)\ \sin \beta l. \tag{5}$$

Messy as they may look, equations (4) and (5) are easily solved since they are equivalent to the matrix equation

$$\begin{pmatrix} -a_0 & \beta \\ a_l\ \cos \beta l - \beta\ \sin \beta l & \beta\ \cos \beta l + a_l \sin \beta l \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{6}$$

Therefore, substituting for D, we have

$$0 = (a_0 C + a_l C)\ \cos \beta l + \left(-\beta C + \frac{a_l a_0 C}{\beta} \right) \sin \beta l. \tag{7}$$

We don't want the trivial solution $C = 0$. We divide by $C \cos \beta l$ and multiply by β to get

$$\boxed{\left(\beta^2 - a_0 a_l\right)\ \tan \beta l = (a_0 + a_l)\beta.} \tag{8}$$

Any root $\beta > 0$ of this "algebraic" equation would give us an eigenvalue $\lambda = \beta^2$.

What would be the corresponding eigenfunction? It would be the above $X(x)$ with the required relation between C and D, namely,

$$\boxed{X(x) = C\left(\cos\beta x + \frac{a_0}{\beta}\sin\beta x\right)} \tag{9}$$

for any $C \neq 0$. By the way, because we divided by $\cos\beta l$, there is the exceptional case when $\cos\beta l = 0$; it would mean by (7) that $\beta = \sqrt{a_0 a_l}$.

Our next task is to solve (8) for β. This is not so easy, as there is no simple formula. One way is to calculate the roots numerically, say by Newton's method. Another way is by graphical analysis, which, instead of precise numerical values, will provide a lot of qualitative information. This is what we'll do. It's here where the nature of a_0 and a_l come into play. Let us rewrite the eigenvalue equation (8) as

$$\tan\beta l = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l}. \tag{10}$$

Our method is to sketch the graphs of the tangent function $y = \tan\beta l$ and the rational function $y = (a_0 + a_l)\beta/(\beta^2 - a_0 a_l)$ as functions of $\beta > 0$ and to find their points of intersection. What the rational function looks like depends on the constants a_0 and a_l.

Case 1 In Figure 1 is pictured the case of *radiation at both ends:* $a_0 > 0$ and $a_l > 0$. Each of the points of intersection (for $\beta > 0$) provides an eigenvalue $\lambda_n = \beta_n^2$. The results depend very much on the a_0 and a_l. The exceptional situation mentioned above, when $\cos\beta l = 0$ and $\beta = \sqrt{a_0 a_l}$, will occur when the graphs of the tangent function and the rational function "intersect at infinity."

No matter what they are, as long as they are both positive, the graph clearly shows that

$$n^2\frac{\pi^2}{l^2} < \lambda_n < (n+1)^2\frac{\pi^2}{l^2} \qquad (n = 0, 1, 2, 3, \ldots). \tag{11}$$

Furthermore,

$$\lim_{n\to\infty} \beta_n - n\frac{\pi}{l} = 0, \tag{12}$$

which means that the larger eigenvalues get relatively closer and closer to $n^2\pi^2/l^2$ (see Exercise 19). You may compare this to the case $a_0 = a_l = 0$, the Neumann problem, where they are all *exactly* equal to $n^2\pi^2/l^2$.

Case 2 The case of absorption at $x = 0$ and radiation at $x = l$, but *more radiation than absorption*, is given by the conditions

$$a_0 < 0, \ a_l > 0, \ a_0 + a_l > 0. \tag{13}$$

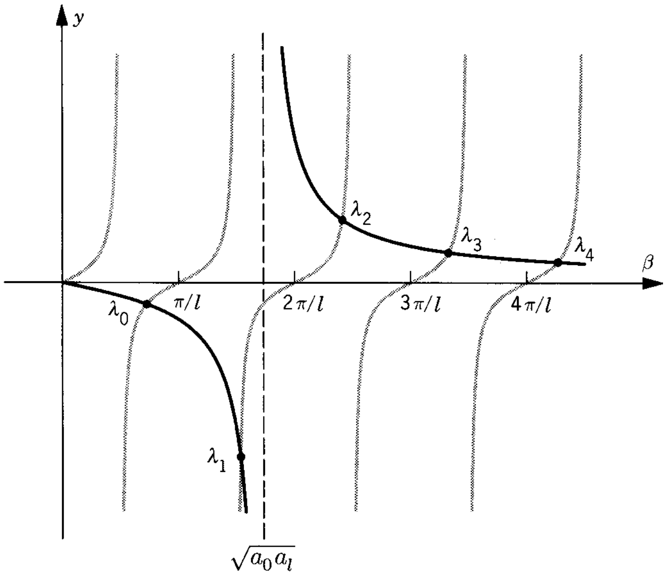

Figure 1

Then the graph looks like Figure 2 or 3, depending on the relative sizes of a_0 and a_l. Once again we see that (11) and (12) hold, except that in Figure 2 there is no eigenvalue λ_0 in the interval $(0,\ \pi^2/l^2)$.

There is an eigenvalue in the interval $(0,\ \pi^2/l^2)$ only if the rational curve crosses the *first* branch of the tangent curve. Since the rational curve has only a single maximum, this crossing can happen only if the slope of the rational curve is greater than the slope of the tangent curve at the origin. Let's

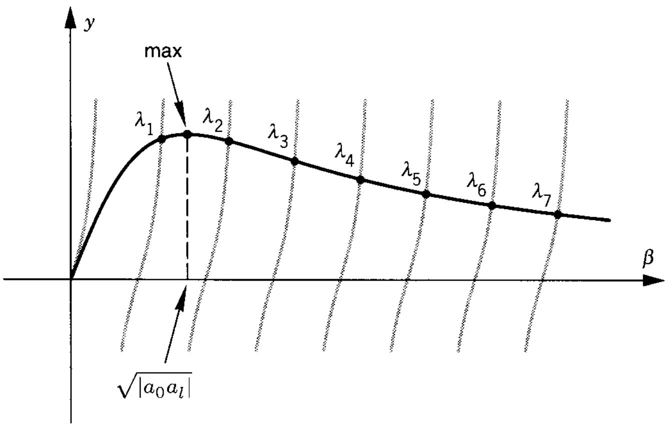

Figure 2

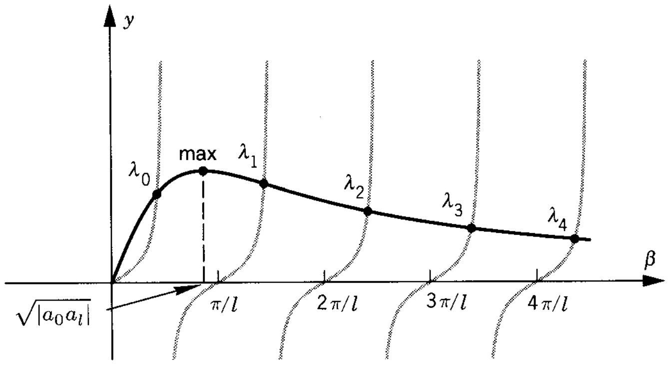

Figure 3

calculate these two slopes. A direct calculation shows that the slope $dy/d\beta$ of the rational curve at the origin is

$$\frac{a_0 + a_l}{-a_0 a_l} = \frac{a_l - |a_0|}{a_l\,|a_0|} > 0$$

because of (13). On the other hand, the slope of the tangent curve $y = \tan l\beta$ at the origin is $l \sec^2(l0) = l$. Thus we reach the following conclusion. In case

$$a_0 + a_l > -a_0 a_l l \tag{14}$$

(which means "much more radiation than absorption"), the rational curve will start out at the origin with a greater slope than the tangent curve and the two graphs must intersect at a point in the interval $(0, \pi/2l)$. Therefore, we conclude that *in Case* 2 *there is an eigenvalue* $0 < \lambda_0 < (\pi/2l)^2$ *if and only if* (14) *holds.*

Other cases, for instance absorption at both ends, may be found in the exercises, especially Exercise 8.

ZERO EIGENVALUE

In Exercise 2 it is shown that *there is a zero eigenvalue if and only if*

$$a_0 + a_l = -a_0 a_l l. \tag{15}$$

Notice that (15) can happen only if a_0 or a_l is negative and the interval has exactly a certain length or else $a_0 = a_l = 0$.

NEGATIVE EIGENVALUE

Now let's investigate the possibility of a negative eigenvalue. This is a very important question; see the discussion at the end of this section. To avoid dealing with imaginary numbers, we set

$$\lambda = -\gamma^2 < 0$$

and write the solution of the differential equation as

$$X(x) = C \cosh \gamma x + D \sinh \gamma x.$$

(An alternative form, which we used at the end of Section 4.1, is $Ae^{\gamma x} + Be^{-\gamma x}$.) The boundary conditions, much as before, lead to the eigenvalue equation

$$\boxed{\tanh \gamma l = -\frac{(a_0 + a_l)\,\gamma}{\gamma^2 + a_0 a_l}.} \tag{16}$$

(Verify it!) So we look for intersections of these two graphs [on the two sides of (16)] for $\gamma > 0$. Any such point of intersection would provide a negative eigenvalue $\lambda = -\gamma^2$ and a corresponding eigenfunction

$$\boxed{X(x) = \cosh \gamma x + \frac{a_0}{\gamma} \sinh \gamma x.} \tag{17}$$

Several different cases are illustrated in Figure 4. Thus in Case 1, of radiation at both ends, when a_0 and a_l are both positive, there is no intersection and so no negative eigenvalue.

Case 2, the situation with more radiation than absorption ($a_0 < 0,\ a_l > 0,\ a_0 + a_l > 0$), is illustrated by the two solid (14) and dashed (18) curves. There is either one intersection or none, depending on the slopes at the origin. The slope of the tanh curve is l, while the slope of the rational curve is

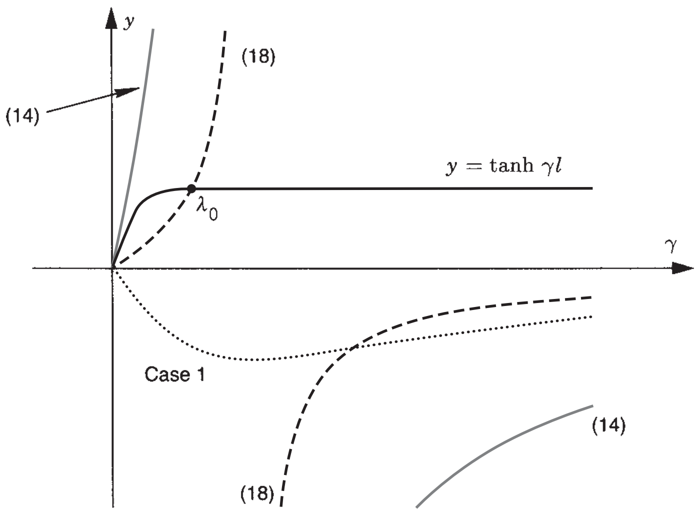

Figure 4

$-(a_0 + a_l)/(a_0 a_1) > 0$. If the last expression is smaller than l, there is an intersection; otherwise, there isn't. So our conclusion in Case 2 is as follows.

Let $a_0 < 0$ *and* $a_l > -a_0$. *If*

$$a_0 + a_l < -a_0 a_l l, \tag{18}$$

then there exists exactly one negative eigenvalue, which we'll call $\lambda_0 < 0$. *If* (14) *holds, then there is no negative eigenvalue*. Notice how the "missing" positive eigenvalue λ_0 in case (18) now makes its appearance as a negative eigenvalue! Furthermore, the zero eigenvalue is the borderline case (15); therefore, we use the notation $\lambda_0 = 0$ in the case of (15).

SUMMARY

We summarize the various cases as follows:

Case 1: Only positive eigenvalues.

Case 2 with (14): Only positive eigenvalues.

Case 2 with (15): Zero is an eigenvalue, all the rest are positive.

Case 2 with (18): One negative eigenvalue, all the rest are positive.

Exercise 8 provides a complete summary of all the other cases.

In any case, that is, for any values for a_0 and a_l, there are no complex, nonreal, eigenvalues. This fact can be shown directly as before but will also be shown by a general, more satisfying, argument in Section 5.3. Furthermore, there are always an infinite number of positive eigenvalues, as is clear from (10). In fact, the tangent function has an infinite number of branches. The rational function on the right side of (10) always goes from the origin to the β axis as $\beta \to \infty$ and so must cross each branch of the tangent except possibly the first one.

For all these problems it is critically important to find *all* the eigenvalues. If even one of them were missing, there would be initial data for which we could not solve the diffusion or wave equations. This will become clearer in Chapter 5. Exactly how we enumerate the eigenvalues, that is, whether we call the first one λ_0 or λ_1 or λ_5 or λ_{-2}, is not important. It is convenient, however, to number them in a consistent way. In the examples presented above we have numbered them in a way that neatly exhibits their dependence on a_0 and a_l.

What Is the Grand Conclusion for the Robin BCs? As before, we have an expansion

$$u(x,t) = \sum_n T_n(t) X_n(x), \tag{19}$$

where $X_n(x)$ are the eigenfunctions and where

$$T_n(t) = \begin{cases} A_n e^{-\lambda_n kt} & \text{for diffusions} \\ A_n \cos(\sqrt{\lambda_n}\, ct) + B_n \sin(\sqrt{\lambda_n}\, ct) & \text{for waves.} \end{cases} \tag{20}$$

Example 1.

Let $a_0 < 0 < a_0 + a_l < -a_0 a_l l$, which is Case 2 with (18). Then the grand conclusion takes the following explicit form. As we showed above, in this case there is exactly one negative eigenvalue $\lambda_0 = -\gamma_0^2 < 0$ as well as a sequence of positive ones $\lambda_n = +\beta_n^2 > 0$ for $n = 1, 2, 3, \ldots$. The complete solution of the diffusion problem

$$\begin{aligned} u_t &= k u_{xx} && \text{for } 0 < x < l, \quad 0 < t < \infty \\ u_x - a_0 u &= 0 && \text{for } x = 0, \quad u_x + a_l u = 0 \quad \text{for } x = l \\ u &= \phi && \text{for } t = 0 \end{aligned}$$

therefore is

$$\boxed{\begin{aligned} u(x,t) = {} & A_0 e^{+\gamma_0^2 kt} \left(\cosh \gamma_0 x + \frac{a_0}{\gamma_0} \sinh \gamma_0 x \right) \\ & + \sum_{n=1}^{\infty} A_n e^{-\beta_n^2 kt} \left(\cos \beta_n x + \frac{a_0}{\beta_n} \sin \beta_n x \right). \end{aligned}} \tag{21}$$

This conclusion (21) has the following physical interpretation if, say, $u(x, t)$ is the *temperature* in a rod of length l. We have taken the case when energy is supplied at $x = 0$ (absorption of energy by the rod, heat flux goes *into* the rod at its left end) and when energy is radiated from the right end (the heat flux goes *out*). For a given length l and a given radiation $a_l > 0$, there is a negative eigenvalue ($\lambda_0 = -\gamma_0^2$) if and only if the absorption is great enough [$|a_0| > a_l/(1 + a_l l)$]. Such a large absorption coefficient allows the temperature to build up to large values, as we see from the expansion (21). In fact, all the terms get smaller as time goes on, except the first one, which *grows* exponentially due to the factor $e^{+\gamma_0^2 kt}$. So the rod gets hotter and hotter (unless $A_0 = 0$, which could only happen for very special initial data).

If, on the other hand, the absorption is relatively small [that is, $|a_0| < a_l/(1 + a_l l)$], then all the eigenvalues are positive and the temperature will remain bounded and will eventually decay to zero. Other interpretations of this sort are left for the exercises. □

For the *wave equation*, a negative eigenvalue $\lambda_0 = -\gamma_0^2$ would also lead to exponential growth because the expansion for $u(x,t)$ would contain the term

$$(A_0 e^{\gamma_0 ct} + B_0 e^{-\gamma_0 ct}) X_0(x).$$

This term comes from the usual equation $-T'' = \lambda c^2 T = -(\gamma_0 c)^2 T$ for the temporal part of a separated solution (see Exercise 10).

EXERCISES

1. Find the eigenvalues graphically for the boundary conditions

$$X(0) = 0, \qquad X'(l) + aX(l) = 0.$$

Assume that $a \neq 0$.

2. Consider the eigenvalue problem with Robin BCs at both ends:

$$-X'' = \lambda X$$
$$X'(0) - a_0 X(0) = 0, \quad X'(l) + a_l X(l) = 0.$$

(a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_l = -a_0 a_l l$.
(b) Find the eigenfunctions corresponding to the zero eigenvalue. (*Hint:* First solve the ODE for *X(x)*. The solutions are not sines or cosines.)

3. Derive the eigenvalue equation (16) for the negative eigenvalues $\lambda = -\gamma^2$ and the formula (17) for the eigenfunctions.
4. Consider the Robin eigenvalue problem. If

$$a_0 < 0, \quad a_l < 0 \quad \text{and} \quad -a_0 - a_l < a_0 a_l l,$$

show that there are *two* negative eigenvalues. This case may be called "substantial absorption at both ends." (*Hint:* Show that the rational curve $y = -(a_0 + a_l)\gamma/(\gamma^2 + a_0 a_l)$ has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places.)

5. In Exercise 4 (substantial absorption at both ends) show graphically that there are an infinite number of positive eigenvalues. Show graphically that they satisfy (11) and (12).
6. If $a_0 = a_l = a$ in the Robin problem, show that:
(a) There are *no* negative eigenvalues if $a \geq 0$, there is *one* if $-2/l < a < 0$, and there are *two* if $a < -2/l$.
(b) Zero is an eigenvalue if and only if $a = 0$ or $a = -2/l$.
7. If $a_0 = a_l = a$, show that as $a \to +\infty$, the eigenvalues tend to the eigenvalues of the Dirichlet problem. That is,

$$\lim_{a \to \infty} \left\{ \beta_n(a) - \frac{(n+1)\pi}{l} \right\} = 0,$$

where $\lambda_n(a) = [\beta_n(a)]^2$ is the $(n+1)$st eigenvalue.

8. Consider again Robin BCs at both ends for arbitrary a_0 and a_l.
   (a) In the $a_0 a_l$ plane sketch the hyperbola $a_0 + a_l = -a_0 a_l l$. Indicate the asymptotes. For (a_0, a_l) on this hyperbola, zero is an eigenvalue, according to Exercise 2(a).
   (b) Show that the hyperbola separates the whole plane into three regions, depending on whether there are two, one, or no negative eigenvalues.
   (c) Label the directions of increasing absorption and radiation on each axis. Label the point corresponding to Neumann BCs.
   (d) Where in the plane do the Dirichlet BCs belong?
9. On the interval $0 \le x \le 1$ of length one, consider the eigenvalue problem

$$-X'' = \lambda X$$
$$X'(0) + X(0) = 0 \qquad \text{and} \qquad X(1) = 0$$

   (absorption at one end and zero at the other).
   (a) Find an eigenfunction with eigenvalue zero. Call it $X_0(x)$.
   (b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.
   (c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.
   (d) Is there a negative eigenvalue?
10. Solve the wave equation with Robin boundary conditions under the assumption that (18) holds.
11. (a) Prove that the (total) energy is conserved for the wave equation with Dirichlet BCs, where the energy is defined to be

$$E = \tfrac{1}{2} \int_0^l \left(c^{-2} u_t^2 + u_x^2\right) dx.$$

   (Compare this definition with Section 2.2.)
   (b) Do the same for the Neumann BCs.
   (c) For the Robin BCs, show that

$$E_R = \tfrac{1}{2} \int_0^l \left(c^{-2} u_t^2 + u_x^2\right) dx + \tfrac{1}{2} a_l [u(l, t)]^2 + \tfrac{1}{2} a_0 [u(0, t)]^2$$

   is conserved. Thus, while the total energy E_R is still a constant, some of the internal energy is "lost" to the boundary if a_0 and a_l are positive and "gained" from the boundary if a_0 and a_l are negative.
12. Consider the unusual eigenvalue problem

$$-v_{xx} = \lambda v \qquad \text{for } 0 < x < l$$

$$v_x(0) = v_x(l) = \frac{v(l) - v(0)}{l}.$$

   (a) Show that $\lambda = 0$ is a double eigenvalue.
   (b) Get an equation for the positive eigenvalues $\lambda > 0$.

(c) Letting $\gamma = \frac{1}{2}l\sqrt{\lambda}$, reduce the equation in part (b) to the equation

$$\gamma \sin\gamma \cos\gamma = \sin^2\gamma.$$

(d) Use part (c) to find half of the eigenvalues explicitly and half of them graphically.
(e) Assuming that all the eigenvalues are nonnegative, make a list of all the eigenfunctions.
(f) Solve the problem $u_t = ku_{xx}$ for $0 < x < l$, with the BCs given above, and with $u(x, 0) = \phi(x)$.
(g) Show that, as $t \to \infty$, $\lim u(x, t) = A + Bx$ for some constants A, B, assuming that you can take limits term by term.

13. Consider a string that is fixed at the end $x = 0$ and is free at the end $x = l$ except that a load (weight) of given mass is attached to the right end.
(a) Show that it satisfies the problem

$$u_{tt} = c^2 u_{xx} \qquad \text{for } 0 < x < l$$

$$u(0, t) = 0 \qquad u_{tt}(l, t) = -ku_x(l, t)$$

for some constant k.
(b) What is the eigenvalue problem in this case?
(c) Find the equation for the positive eigenvalues and find the eigenfunctions.

14. Solve the eigenvalue problem $x^2u'' + 3xu' + \lambda u = 0$ for $1 < x < e$, with $u(1) = u(e) = 0$. Assume that $\lambda > 1$. (*Hint:* Look for solutions of the form $u = x^m$.)

15. Find the equation for the eigenvalues λ of the problem

$$(\kappa(x)X')' + \lambda\rho(x)X = 0 \qquad \text{for } 0 < x < l \text{ with } X(0) = X(l) = 0,$$

where $\kappa(x) = \kappa_1^2$ for $x < a$, $\kappa(x) = \kappa_2^2$ for $x > a$, $\rho(x) = \rho_1^2$ for $x < a$, and $\rho(x) = \rho_2^2$ for $x > a$. All these constants are positive and $0 < a < l$.

16. Find the positive eigenvalues and the corresponding eigenfunctions of the fourth-order operator $+d^4/dx^4$ with the four boundary conditions

$$X(0) = X(l) = X''(0) = X''(l) = 0.$$

17. Solve the fourth-order eigenvalue problem $X'''' = \lambda X$ in $0 < x < l$, with the four boundary conditions

$$X(0) = X'(0) = X(l) = X'(l) = 0,$$

where $\lambda > 0$. (*Hint:* First solve the fourth-order ODE.)

18. A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. Each such bar is clamped at one end and is approximately modeled by the fourth-order PDE $u_{tt} + c^2u_{xxxx} = 0$. It has initial conditions as for the wave equation. Let's say that on the end $x = 0$ it is clamped (fixed), meaning that it satisfies

$u(0, t) = u_x(0, t) = 0$. On the other end $x = l$ it is free, meaning that it satisfies $u_{xx}(l, t) = u_{xxx}(l, t) = 0$. Thus there are a total of four boundary conditions, two at each end.

(a) Separate the time and space variables to get the eigenvalue problem $X'''' = \lambda X$.
(b) Show that zero is not an eigenvalue.
(c) Assuming that all the eigenvalues are positive, write them as $\lambda = \beta^4$ and find the equation for β.
(d) Find the frequencies of vibration.
(e) Compare your answer in part (d) with the overtones of the vibrating string by looking at the ratio β_2^2/β_1^2. Explain why you hear an almost pure tone when you listen to a tuning fork.

19. Show that in Case 1 (radiation at both ends)

$$\lim_{n\to\infty} \left[\lambda_n - \frac{n^2\pi^2}{l^2} \right] = \frac{2}{l}\,(a_0 + a_l)\,.$$