2

WAVES AND DIFFUSIONS

In this chapter we study the wave and diffusion equations on the whole real line $-\infty < x < +\infty$. Real physical situations are usually on finite intervals. We are justified in taking x on the whole real line for two reasons. Physically speaking, if you are sitting far away from the boundary, it will take a certain time for the boundary to have a substantial effect on you, and until that time the solutions we obtain in this chapter are valid. Mathematically speaking, the absence of a boundary is a big simplification. The most fundamental properties of the PDEs can be found most easily without the complications of boundary conditions. That is the purpose of this chapter. We begin with the wave equation.

2.1 THE WAVE EQUATION

We write the wave equation as

$$\boxed{u_{tt} = c^2 u_{xx} \qquad \text{for } -\infty < x < +\infty.} \tag{1}$$

(Physically, you can imagine a very long string.) This is the simplest second-order equation. The reason is that the operator factors nicely:

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0. \tag{2}$$

This means that, starting from a function $u(x, t)$, you compute $u_t + cu_x$, call the result v, then you compute $v_t - cv_x$, and you ought to get the zero function. The *general solution* is

$$\boxed{u(x, t) = f(x + ct) + g(x - ct)} \tag{3}$$

where f and g are two *arbitrary* (twice differentiable) functions of a single variable.

Proof. Because of (2), if we let $v = u_t + cu_x$, we must have $v_t - cv_x = 0$. Thus we have two first-order equations

$$v_t - cv_x = 0 \tag{4a}$$

and

$$u_t + cu_x = v. \tag{4b}$$

These two first-order equations are equivalent to (1) itself. Let's solve them one at a time. As we know from Section 1.2, equation (4a) has the solution $v(x, t) = h(x + ct)$, where h is any function.

So we must solve the other equation, which now takes the form

$$u_t + cu_x = h(x + ct) \tag{4c}$$

for the unknown function $u(x, t)$. It is easy to check directly by differentiation that one solution is $u(x, t) = f(x + ct)$, where $f'(s) = h(s)/2c$. [A prime ($'$) denotes the derivative of a function of one variable.] To the solution $f(x + ct)$ we can add $g(x - ct)$ to get another solution (since the equation is linear). The most general solution of (4b) in fact turns out to be a particular solution plus any solution of the homogeneous equation; that is,

$$u(x, t) = f(x + ct) + g(x - ct),$$

as asserted by the theorem. The complete justification is left to be worked out in Exercise 4.

A different method to derive the solution formula (3) is to introduce the *characteristic coordinates*

$$\xi = x + ct \qquad \eta = x - ct.$$

By the chain rule, we have $\partial_x = \partial_\xi + \partial_\eta$ and $\partial_t = c\partial_\xi + c\partial_\eta$. Therefore, $\partial_t - c\partial_x = -2c\partial_\eta$ and $\partial_t + c\partial_x = 2c\partial_\xi$. So equation (1) takes the form

$$(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = (-2c\partial_\xi)(2c\partial_\eta)u = 0,$$

which means that $u_{\xi\eta} = 0$ since $c \neq 0$. The solution of this transformed equation is

$$u = f(\xi) + g(\eta)$$

(see Section 1.1), which agrees exactly with the previous answer (3). □

The wave equation has a nice simple geometry. There are *two* families of characteristic lines, $x \pm ct =$ constant, as indicated in Figure 1. The most general solution is the sum of two functions. One, $g(x - ct)$, is a wave of arbitrary shape traveling to the *right* at speed c. The other, $f(x + ct)$, is another shape traveling to the *left* at speed c. A "movie" of $g(x - ct)$ is sketched in Figure 1 of Section 1.3.

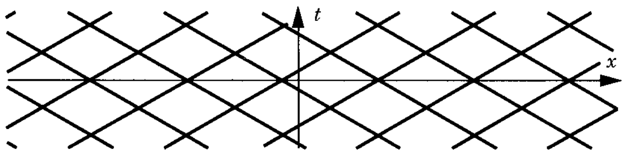

Figure 1

INITIAL VALUE PROBLEM

The initial-value problem is to solve the wave equation

$$\boxed{u_{tt} = c^2 u_{xx} \qquad \text{for } -\infty < x < +\infty} \tag{1}$$

with the initial conditions

$$\boxed{u(x, 0) = \phi(x) \qquad u_t(x, 0) = \psi(x),} \tag{5}$$

where ϕ and ψ are arbitrary functions of x. There is one, and only one, solution of this problem. For instance, if $\phi(x) = \sin x$ and $\psi(x) = 0$, then $u(x, t) = \sin x \cos ct$.

The solution of (1),(5) is easily found from the general formula (3). First, setting $t = 0$ in (3), we get

$$\phi(x) = f(x) + g(x). \tag{6}$$

Then, using the chain rule, we differentiate (3) with respect to t and put $t = 0$ to get

$$\psi(x) = cf'(x) - cg'(x). \tag{7}$$

Let's regard (6) and (7) as two equations for the two unknown functions f and g. To solve them, it is convenient temporarily to change the name of the variable to some neutral name; we change the name of x to s. Now we differentiate (6) and divide (7) by c to get

$$\phi' = f' + g' \qquad \text{and} \qquad \frac{1}{c}\psi = f' - g'.$$

Adding and subtracting the last pair of equations gives us

$$f' = \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \qquad \text{and} \qquad g' = \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right).$$

Integrating, we get

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A$$

and

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B,$$

where A and B are constants. Because of (6), we have $A + B = 0$. This tells us what f and g are in the general formula (3). Substituting $s = x + ct$ into the formula for f and $s = x - ct$ into that of g, we get

$$u(x, t) = \frac{1}{2}\phi(x + ct) + \frac{1}{2c}\int_0^{x+ct} \psi + \frac{1}{2}\phi(x - ct) - \frac{1}{2c}\int_0^{x-ct} \psi.$$

This simplifies to

$$\boxed{u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.} \tag{8}$$

This is the solution formula for the initial-value problem, due to d'Alembert in 1746. Assuming ϕ to have a continuous second derivative (written $\phi \in C^2$) and ψ to have a continuous first derivative ($\psi \in C^1$), we see from (8) that u itself has continuous second partial derivatives in x and t ($u \in C^2$). Then (8) is a *bona fide* solution of (1) and (5). You may check this directly by differentiation and by setting $t = 0$.

Example 1.

For $\phi(x) \equiv 0$ and $\psi(x) = \cos x$, the solution is $u(x, t) = (1/2c)$ $[\sin(x + ct) - \sin(x - ct)] = (1/c)\cos x \sin ct$. Checking this result directly, we have $u_{tt} = -c\cos x \sin ct$, $u_{xx} = -(1/c)\cos x \sin ct$, so that $u_{tt} = c^2 u_{xx}$. The initial condition is easily checked. □

Example 2. The Plucked String

For a vibrating string the speed is $c = \sqrt{T/\rho}$. Consider an infinitely long string with initial position

$$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a} & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases} \tag{9}$$

and initial velocity $\psi(x) \equiv 0$ for all x. This is a "three-finger" pluck, with all three fingers removed at once. A "movie" of this solution $u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)]$ is shown in Figure 2. (Even though this solution is not twice differentiable, it can be shown to be a "weak" solution, as discussed later in Section 12.1.)

Each of these pictures is the sum of two triangle functions, one moving to the right and one to the left, as is clear graphically. To write down the formulas that correspond to the pictures requires a lot more work. The formulas depend on the relationships among the five numbers $0, \pm a, x \pm ct$. For instance, let $t = a/2c$. Then $x \pm ct = x \pm a/2$. First, if $x < -3a/2$, then $x \pm a/2 < -a$ and $u(x, t) \equiv 0$. Second, if $-3a/2 < x < -a/2$, then

$$u(x,t) = \frac{1}{2}\phi\left(x + \frac{1}{2}a\right) = \frac{1}{2}\left(b - \frac{b|x + \frac{1}{2}a|}{a}\right) = \frac{3b}{4} + \frac{bx}{2a}.$$

Third, if $|x| < a/2$, then

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left[\phi\left(x + \frac{1}{2}a\right) + \phi\left(x - \frac{1}{2}a\right)\right] \\ &= \frac{1}{2}\left[b - \frac{b\left(x + \frac{1}{2}a\right)}{a} + b - \frac{b\left(\frac{1}{2}a - x\right)}{a}\right] \\ &= \frac{1}{2}b \end{aligned}$$

and so on [see Figure 2]. □

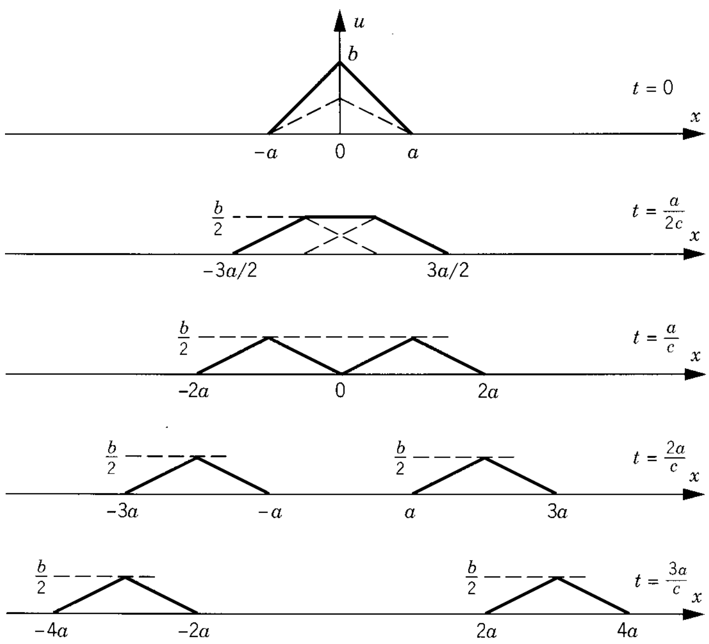

Figure 2

EXERCISES

1. Solve $u_{tt} = c^2 u_{xx},\ u(x, 0) = e^x, u_t(x, 0) = \sin x$.
2. Solve $u_{tt} = c^2 u_{xx},\ u(x, 0) = \log(1 + x^2), u_t(x, 0) = 4 + x$.
3. The midpoint of a piano string of tension T, density ρ, and length l is hit by a hammer whose head diameter is $2a$. A flea is sitting at a distance $l/4$ from one end. (Assume that $a < l/4$; otherwise, poor flea!) How long does it take for the disturbance to reach the flea?
4. Justify the conclusion at the beginning of Section 2.1 that *every* solution of the wave equation has the form $f(x + ct) + g(x - ct)$.
5. (*The hammer blow*) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c,\ a/c,\ 3a/2c,\ 2a/c$, and $5a/c$. [*Hint:* Calculate

$$u(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds = \frac{1}{2c}\{\text{length of } (x - ct, x + ct) \cap (-a, a)\}.$$

Then $u(x, a/2c) = (1/2c)$ {length of $(x - a/2,\ x + a/2) \cap (-a,\ a)$}. This takes on different values for $|x| < a/2$, for $a/2 < x < 3a/2$, and for $x > 3a/2$. Continue in this manner for each case.]

6. In Exercise 5, find the greatest displacement, $\max_x u(x, t)$, as a function of t.
7. If both ϕ and ψ are odd functions of x, show that the solution $u(x, t)$ of the wave equation is also odd in x for all t.
8. A *spherical wave* is a solution of the three-dimensional wave equation of the form $u(r, t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \qquad (\text{“spherical wave equation”}).$$

   (a) Change variables $v = ru$ to get the equation for v: $v_{tt} = c^2 v_{rr}$.
   (b) Solve for v using (3) and thereby solve the spherical wave equation.
   (c) Use (8) to solve it with initial conditions $u(r, 0) = \phi(r)$, $u_t(r, 0) = \psi(r)$, taking both $\phi(r)$ and $\psi(r)$ to be even functions of r.

9. Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0,\ u(x, 0) = x^2,\ u_t(x, 0) = e^x$. (*Hint:* Factor the operator as we did for the wave equation.)
10. Solve $u_{xx} + u_{xt} - 20u_{tt} = 0, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x)$.
11. Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$.

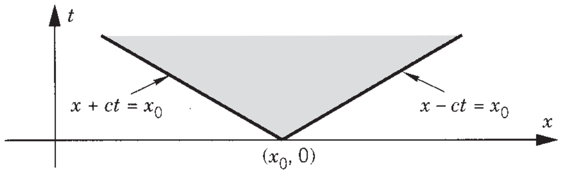

Figure 1

2.2 CAUSALITY AND ENERGY

CAUSALITY

We have just learned that the effect of an initial position $\phi(x)$ is a pair of waves traveling in either direction at speed c and at half the original amplitude. The effect of an initial velocity ψ is a wave spreading out at speed $\leq c$ in both directions (see Exercise 2.1.5 for an example). So part of the wave may lag behind (if there is an initial velocity), but *no part goes faster than speed c*. The last assertion is called the *principle of causality.* It can be visualized in the xt plane in Figure 1.

An initial condition (position or velocity or both) at the point $(x_0, 0)$ can affect the solution for $t > 0$ only in the shaded sector, which is called the *domain of influence* of the point $(x_0, 0)$. As a consequence, if ϕ and ψ vanish for $|x| > R$, then $u(x, t) = 0$ for $|x| > R + ct$. In words, the domain of influence of an interval ($|x| \leq R$) is a sector ($|x| \leq R + ct$).

An "inverse" way to express causality is the following. *Fix* a point (x, t) for $t > 0$ (see Figure 2). How is the number $u(x, t)$ synthesized from the initial data ϕ, ψ? It depends only on the values of ϕ at the two points $x \pm ct$, and it depends only on the values of ψ within the interval $[x - ct, x + ct]$. We therefore say that the interval $(x - ct, x + ct)$ is the interval of dependence of the point (x, t) on $t = 0$. Sometimes we call the entire shaded triangle Δ the *domain of dependence* or the *past history* of the point (x, t). The domain of dependence is bounded by the pair of characteristic lines that pass through (x, t).

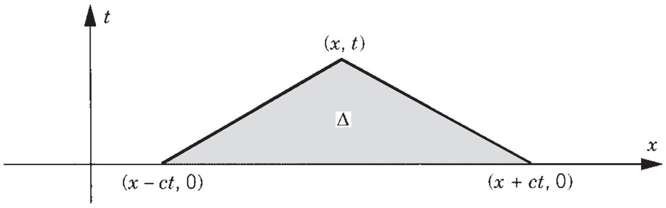

Figure 2

ENERGY

Imagine an infinite string with constants ρ and T. Then $\rho u_{tt} = T u_{xx}$ for $-\infty < x < +\infty$. From physics we know that the kinetic energy is $\frac{1}{2}mv^2$, which in our case takes the form $\text{KE} = \frac{1}{2}\rho \int u_t^2 \, dx$. This integral, and the following ones, are evaluated from $-\infty$ to $+\infty$. To be sure that the integral converges, we assume that $\phi(x)$ and $\psi(x)$ vanish outside an interval $\{|x| \leq R\}$. As mentioned above, $u(x, t)$ [and therefore $u_t(x, t)$] vanish for $|x| > R + ct$. Differentiating the kinetic energy, we can pass the derivative under the integral sign (see Section A.3) to get

$$\frac{dKE}{dt} = \rho \int u_t u_{tt} \, dx.$$

Then we substitute the PDE $\rho u_{tt} = T u_{xx}$ and integrate by parts to get

$$\frac{dKE}{dt} = T \int u_t u_{xx} \, dx = T u_t u_x - T \int u_{tx} u_x \, dx.$$

The term $Tu_t u_x$ is evaluated at $x = \pm\infty$ and so it vanishes. But the final term is a pure derivative since $u_{tx} u_x = (\frac{1}{2}u_x^2)_t$. Therefore,

$$\frac{dKE}{dt} = -\frac{d}{dt} \int \frac{1}{2} T u_x^2 \, dx.$$

Let $PE = \frac{1}{2} T \int u_x^2 \, dx$ and let $E = KE + PE$. Then $dKE/dt = -dPE/dt$, or $dE/dt = 0$. Thus

$$\boxed{E = \tfrac{1}{2} \int_{-\infty}^{+\infty} \left(\rho u_t^2 + T u_x^2\right) dx} \tag{1}$$

is a constant independent of t. This is the law of *conservation of energy.*

In physics courses we learn that PE has the interpretation of the potential energy. The only thing we need mathematically is the total energy E. The conservation of energy is one of the most basic facts about the wave equation. Sometimes the definition of E is modified by a constant factor, but that does not affect its conservation. Notice that the energy is necessarily positive. The energy can also be used to derive causality (as will be done in Section 9.1).

Example 1.

The plucked string, Example 2 of Section 2.1, has the energy

$$E = \frac{1}{2} T \int \phi_x^2 \, dx = \frac{1}{2} T \left(\frac{b}{a}\right)^2 2a = \frac{Tb^2}{a}. \qquad \square$$

In electromagnetic theory the equations are Maxwell's. Each component of the electric and magnetic fields satisfies the (three-dimensional) wave equation, where c is the speed of light. The principle of causality, discussed above, is the cornerstone of the theory of relativity. It means that a signal located at the position x_0 at the instant t_0 cannot move faster than the speed of light. The domain of influence of (x_0, t_0) consists of all the points that can be reached by a signal of speed c starting from the point x_0 at the time t_0. It turns out that the solutions of the *three*-dimensional wave equation always travel at speeds exactly equal to c and never slower. Therefore, the causality principle is sharper in three dimensions than in one. This sharp form is called *Huygens's principle* (see Chapter 9).

Flatland is an imaginary two-dimensional world. You can think of yourself as a waterbug confined to the surface of a pond. You wouldn't want to live there because Huygens's principle is not valid in two dimensions (see Section 9.2). Each sound you make would automatically mix with the "echoes" of your previous sounds. And each view would be mixed fuzzily with the previous views. Three is the best of all possible dimensions.

EXERCISES

1. Use the energy conservation of the wave equation to prove that the only solution with $\phi \equiv 0$ and $\psi \equiv 0$ is $u \equiv 0$. (*Hint:* Use the first vanishing theorem in Section A.1.)
2. For a solution $u(x, t)$ of the wave equation with $\rho = T = c = 1$, the energy density is defined as $e = \frac{1}{2}(u_t^2 + u_x^2)$ and the momentum density as $p = u_t u_x$.
   (a) Show that $\partial e/\partial t = \partial p/\partial x$ and $\partial p/\partial t = \partial e/\partial x$.
   (b) Show that both $e(x, t)$ and $p(x, t)$ also satisfy the wave equation.
3. Show that the wave equation has the following invariance properties.
   (a) Any translate $u(x - y, t)$, where y is fixed, is also a solution.
   (b) Any derivative, say u_x, of a solution is also a solution.
   (c) The dilated function $u(ax, at)$ is also a solution, for any constant a.
4. If $u(x, t)$ satisfies the wave equation $u_{tt} = u_{xx}$, prove the identity

   $$u(x + h,\ t + k) + u(x - h,\ t - k) = u(x + k,\ t + h) + u(x - k,\ t - h)$$

   for all x, t, h, and k. Sketch the quadrilateral Q whose vertices are the arguments in the identity.
5. For the *damped* string, equation (1.3.3), show that the energy decreases.
6. Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in n-dimensional space satisfies the PDE

   $$u_{tt} = c^2\left(u_{rr} + \frac{n-1}{r}u_r\right),$$

   where r is the spherical coordinate. Consider such a wave that has the special form $u(r, t) = \alpha(r) f(t - \beta(r))$, where $\alpha(r)$ is called the attenuation and $\beta(r)$ the delay. The question is whether such solutions exist for "arbitrary" functions f.

(a) Plug the special form into the PDE to get an ODE for f.
(b) Set the coefficients of f'', f', and f equal to zero.
(c) Solve the ODEs to see that $n = 1$ or $n = 3$ (unless $u \equiv 0$).
(d) If $n = 1$, show that $\alpha(r)$ is a constant (so that "there is no attenuation").

(T. Morley, *American Mathematical Monthly*, Vol. 27, pp. 69–71, 1985)

2.3 THE DIFFUSION EQUATION

In this section we begin a study of the one-dimensional diffusion equation

$$u_t = ku_{xx}. \tag{1}$$

Diffusions are very different from waves, and this is reflected in the mathematical properties of the equations. Because (1) is harder to solve than the wave equation, we begin this section with a general discussion of some of the properties of diffusions. We begin with the maximum principle, from which we'll deduce the uniqueness of an initial-boundary problem. We postpone until the next section the derivation of the solution formula for (1) on the whole real line.

Maximum Principle. If $u(x, t)$ satisfies the diffusion equation in a rectangle (say, $0 \le x \le l,\ 0 \le t \le T$) in space-time, then the maximum value of $u(x, t)$ is assumed either initially ($t = 0$) or on the lateral sides ($x = 0$ or $x = l$) (see Figure 1).

In fact, there is a *stronger version* of the maximum principle which asserts that the maximum cannot be assumed anywhere inside the rectangle but *only on the bottom or the lateral sides* (unless u is a constant). The corners are allowed.

The minimum value has the same property; it too can be attained only on the bottom or the lateral sides. To prove the minimum principle, just apply the maximum principle to $[-u(x, t)]$.

These principles have a natural interpretation in terms of diffusion or heat flow. If you have a rod with no internal heat source, the hottest spot and the

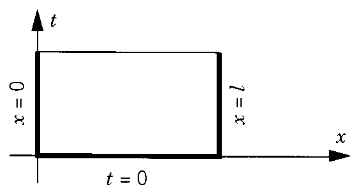

Figure 1

coldest spot can occur only initially or at one of the two ends of the rod. Thus a hot spot at time zero will cool off (unless heat is fed into the rod at an end). You can burn one of its ends but the maximum temperature will always be at the hot end, so that it will be cooler away from that end. Similarly, if you have a substance diffusing along a tube, its highest concentration can occur only initially or at one of the ends of the tube.

If we draw a "movie" of the solution, the maximum drops down while the minimum comes up. So the differential equation tends to smooth the solution out. (This is very different from the behavior of the wave equation!)

Proof of the Maximum Principle. We'll prove only the weaker version. (Surprisingly, its strong form is much more difficult to prove.) For the strong version, see [PW]. The idea of the proof is to use the fact, from calculus, that at an interior maximum the first derivatives vanish and the second derivatives satisfy inequalities such as $u_{xx} \le 0$. If we knew that $u_{xx} \ne 0$ at the maximum (which we do not), then we'd have $u_{xx} < 0$ as well as $u_t = 0$, so that $u_t \ne k u_{xx}$. This contradiction would show that the maximum could only be somewhere on the boundary of the rectangle. However, because u_{xx} could in fact be equal to zero, we need to play a mathematical game to make the argument work.

So let M denote the maximum value of $u(x, t)$ on the three sides $t = 0$, $x = 0$, and $x = l$. (Recall that any continuous function on any bounded closed set is bounded and assumes its maximum on that set.) We must show that $u(x, t) \le M$ throughout the rectangle R.

Let ϵ be a positive constant and let $v(x, t) = u(x, t) + \epsilon x^2$. Our goal is to show that $v(x, t) \le M + \epsilon l^2$ throughout R. Once this is accomplished, we'll have $u(x, t) \le M + \epsilon(l^2 - x^2)$. This conclusion is true for any $\epsilon > 0$. Therefore, $u(x, t) \le M$ throughout R, which is what we are trying to prove.

Now from the definition of v, it is clear that $v(x, t) \le M + \epsilon l^2$ on $t = 0$, on $x = 0$, and on $x = l$. This function v satisfies

$$v_t - k v_{xx} = u_t - k(u + \epsilon x^2)_{xx} = u_t - k u_{xx} - 2\epsilon k = -2\epsilon k < 0, \qquad (2)$$

which is the "diffusion inequality." Now suppose that $v(x, t)$ attains its maximum at an *interior* point (x_0, t_0). That is, $0 < x_0 < l$, $0 < t_0 < T$. By ordinary calculus, we know that $v_t = 0$ and $v_{xx} \le 0$ at (x_0, t_0). This contradicts the diffusion inequality (2). So there can't be an interior maximum. Suppose now that $v(x, t)$ has a maximum (in the closed rectangle) at a point on the *top* edge $\{t_0 = T \text{ and } 0 < x < l\}$. Then $v_x(x_0, t_0) = 0$ and $v_{xx}(x_0, t_0) \le 0$, as before. Furthermore, because $v(x_0, t_0)$ is bigger than $v(x_0, t_0 - \delta)$, we have

$$v_t(x_0, t_0) = \lim \frac{v(x_0, t_0) - v(x_0, t_0 - \delta)}{\delta} \ge 0$$

as $\delta \to 0$ through positive values. (This is not an equality because the maximum is only "one-sided" in the variable t.) We again reach a contradiction to the diffusion inequality.

But $v(x,t)$ does have a maximum *somewhere* in the closed rectangle $0 \le x \le l$, $0 \le t \le T$. This maximum must be on the bottom or sides. Therefore $v(x, t) \le M + \epsilon l^2$ throughout R. This proves the maximum principle (in its weaker version).

UNIQUENESS

The maximum principle can be used to give a proof of *uniqueness for the Dirichlet problem for the diffusion equation*. That is, there is at most one solution of

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad \text{for } 0 < x < l \text{ and } t > 0 \\ u(x,0) &= \phi(x) \\ u(0,t) &= g(t) \qquad u(l,t) = h(t) \end{aligned} \tag{3}$$

for four given functions f, ϕ, g, and h. Uniqueness means that any solution is determined completely by its initial and boundary conditions. Indeed, let $u_1(x, t)$ and $u_2(x, t)$ be two solutions of (3). Let $w = u_1 - u_2$ be their difference. Then $w_t - kw_{xx} = 0$, $w(x,0) = 0$, $w(0,t) = 0$, $w(l,t) = 0$. Let $T > 0$. By the maximum principle, $w(x,t)$ has its maximum for the rectangle on its bottom or sides—exactly where it vanishes. So $w(x,t) \le 0$. The same type of argument for the minimum shows that $w(x,t) \ge 0$. Therefore, $w(x,t) \equiv 0$, so that $u_1(x,t) \equiv u_2(x,t)$ for all $t \ge 0$.

Here is a second proof of uniqueness for problem (3), by a very different technique, the *energy method*. Multiplying the equation for $w = u_1 - u_2$ by w itself, we can write

$$0 = 0 \cdot w = (w_t - kw_{xx})(w) = \left(\tfrac{1}{2}w^2\right)_t + (-kw_x w)_x + kw_x^2.$$

(Verify this by carrying out the derivatives on the right side.) Upon integrating over the interval $0 < x < l$, we get

$$0 = \int_0^l \left(\tfrac{1}{2}w^2\right)_t dx - kw_x w\Big|_{x=0}^{x=l} + k\int_0^l w_x^2\, dx.$$

Because of the boundary conditions ($w = 0$ at $x = 0, l$),

$$\frac{d}{dt}\int_0^l \frac{1}{2}[w(x,t)]^2\, dx = -k\int_0^l [w_x(x,t)]^2\, dx \le 0,$$

where the time derivative has been pulled out of the x integral (see Section A.3). Therefore, $\int w^2\, dx$ is decreasing, so

$$\int_0^l [w(x,t)]^2\, dx \le \int_0^l [w(x,0)]^2\, dx \tag{4}$$

for $t \ge 0$. The right side of (4) vanishes because the initial conditions of u and v are the same, so that $\int [w(x,t)]^2\, dx = 0$ for all $t > 0$. So $w \equiv 0$ and $u_1 \equiv u_2$ for all $t \ge 0$.

STABILITY

This is the third ingredient of well-posedness (see Section 1.5). It means that the initial and boundary conditions are correctly formulated. The energy method leads to the following form of stability of problem (3), in case $h = g = f = 0$. Let $u_1(x, 0) = \phi_1(x)$ and $u_2(x, 0) = \phi_2(x)$. Then $w = u_1 - u_2$ is the solution with the initial datum $\phi_1 - \phi_2$. So from (4) we have

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2\,dx \le \int_0^l [\phi_1(x) - \phi_2(x)]^2\,dx. \qquad (5)$$

On the right side is a quantity that measures the nearness of the initial data for two solutions, and on the left we measure the nearness of the solutions at any later time. Thus, if we start nearby (at $t = 0$), we stay nearby. This is exactly the meaning of stability in the "square integral" sense (see Sections 1.5 and 5.4).

The maximum principle also proves the stability, but with a different way to measure nearness. Consider two solutions of (3) in a rectangle. We then have $w \equiv u_1 - u_2 = 0$ on the lateral sides of the rectangle and $w = \phi_1 - \phi_2$ on the bottom. The maximum principle asserts that throughout the rectangle

$$u_1(x,t) - u_2(x,t) \le \max|\phi_1 - \phi_2|.$$

The "minimum" principle says that

$$u_1(x,t) - u_2(x,t) \ge -\max|\phi_1 - \phi_2|.$$

Therefore,

$$\max_{0\le x\le l} |u_1(x,t) - u_2(x,t)| \le \max_{0\le x\le l} |\phi_1(x) - \phi_2(x)|, \qquad (6)$$

valid for all $t > 0$. Equation (6) is in the same spirit as (5), but with a quite different method of measuring the nearness of functions. It is called stability in the "uniform" sense.

EXERCISES

1. Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and its minimum in the closed rectangle $\{0 \le x \le 1,\ 0 \le t \le T\}$.
2. Consider a solution of the diffusion equation $u_t = u_{xx}$ in $\{0 \le x \le l,\ 0 \le t < \infty\}$.
   (a) Let $M(T)$ = the maximum of $u(x, t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $M(T)$ increase or decrease as a function of T?
   (b) Let $m(T)$ = the minimum of $u(x, t)$ in the closed rectangle $\{0 \le x \le l, 0 \le t \le T\}$. Does $m(T)$ increase or decrease as a function of T?
3. Consider the diffusion equation $u_t = u_{xx}$ in the interval (0, 1) with $u(0, t) = u(1, t) = 0$ and $u(x, 0) = 1 - x^2$. Note that this initial function does not satisfy the boundary condition at the left end, but that the solution will satisfy it for all $t > 0$.

(a) Show that $u(x, t) > 0$ at all interior points $0 < x < 1,\ 0 < t < \infty$.
(b) For each $t > 0$, let $\mu(t) =$ the maximum of $u(x, t)$ over $0 \le x \le 1$. Show that $\mu(t)$ is a decreasing (i.e., nonincreasing) function of t. (*Hint:* Let the maximum occur at the point $X(t)$, so that $\mu(t) = u(X(t), t)$. Differentiate $\mu(t)$, assuming that $X(t)$ is differentiable.)
(c) Draw a rough sketch of what you think the solution looks like (u versus x) at a few times. (If you have appropriate software available, compute it.)

4. Consider the diffusion equation $u_t = u_{xx}$ in $\{0 < x < 1,\ 0 < t < \infty\}$ with $u(0,\ t) = u(1, t) = 0$ and $u(x, 0) = 4x(1 - x)$.
(a) Show that $0 < u(x, t) < 1$ for all $t > 0$ and $0 < x < 1$.
(b) Show that $u(x, t) = u(1 - x, t)$ for all $t \ge 0$ and $0 \le x \le 1$.
(c) Use the energy method to show that $\int_0^1 u^2\,dx$ is a strictly decreasing function of t.

5. The purpose of this exercise is to show that the maximum principle is not true for the equation $u_t = xu_{xx}$, which has a variable coefficient.
(a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \le x \le 2,\ 0 \le t \le 1\}$.
(b) Where precisely does our proof of the maximum principle break down for this equation?

6. Prove the *comparison principle* for the diffusion equation: If u and v are two solutions, and if $u \le v$ for $t = 0$, for $x = 0$, and for $x = l$, then $u \le v$ for $0 \le t < \infty,\ 0 \le x \le l$.

7. (a) More generally, if $u_t - ku_{xx} = f,\ v_t - kv_{xx} = g,\ f \le g$, and $u \le v$ at $x = 0$, $x = l$ and $t = 0$, prove that $u \le v$ for $0 \le x \le l$, $0 \le t < \infty$.
(b) If $v_t - v_{xx} \ge \sin x$ for $0 \le x \le \pi,\ 0 < t < \infty$, and if $v(0, t) \ge 0$, $v(\pi, t) \ge 0$ and $v(x, 0) \ge \sin x$, use part (a) to show that $v(x, t) \ge (1 - e^{-t}) \sin x$.

8. Consider the diffusion equation on $(0, l)$ with the Robin boundary conditions $u_x(0, t) - a_0 u(0, t) = 0$ and $u_x(l, t) + a_l u(l, t) = 0$. If $a_0 > 0$ and $a_l > 0$, use the energy method to show that the endpoints contribute to the decrease of $\int_0^l u^2(x,\ t)\,dx$. (This is interpreted to mean that part of the "energy" is lost at the boundary, so we call the boundary conditions "*radiating*" or "*dissipative*.")

2.4 DIFFUSION ON THE WHOLE LINE

Our purpose in this section is to solve the problem

$$\boxed{\begin{aligned} u_t &= ku_{xx} \quad (-\infty < x < \infty,\ 0 < t < \infty) \qquad &(1)\\ u(x, 0) &= \phi(x). &(2)\end{aligned}}$$

As with the wave equation, the problem on the infinite line has a certain "purity", which makes it easier to solve than the finite-interval problem. (The effects of boundaries will be discussed in the next several chapters.) Also as with the wave equation, we will end up with an explicit formula. But it will be derived by a method *very different* from the methods used before. (The characteristics for the diffusion equation are just the lines $t =$ constant and play no major role in the analysis.) Because the solution of (1) is not easy to derive, we first set the stage by making some general comments.

Our method is to solve it for a *particular* $\phi(x)$ and then build the general solution from this particular one. We'll use five basic *invariance properties* of the diffusion equation (1).

(a) The *translate* $u(x-y, t)$ of any solution $u(x, t)$ is another solution, for any fixed y.

(b) Any *derivative* (u_x or u_t or u_{xx}, etc.) of a solution is again a solution.

(c) A *linear combination* of solutions of (1) is again a solution of (1). (This is just linearity.)

(d) An *integral* of solutions is again a solution. Thus if $S(x, t)$ is a solution of (1), then so is $S(x-y, t)$ and so is

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)\,dy$$

for any function $g(y)$, as long as this improper integral converges appropriately. (We'll worry about convergence later.) In fact, (d) is just a limiting form of (c).

(e) If $u(x,t)$ is a solution of (1), so is the *dilated* function $u(\sqrt{a}\,x, at)$, for any $a > 0$. Prove this by the chain rule: Let $v(x,t) = u(\sqrt{a}\,x, at)$. Then $v_t = [\partial(at)/\partial t]u_t = au_t$ and $v_x = [\partial(\sqrt{a}\,x)/\partial x]u_x = \sqrt{a}\,u_x$ and $v_{xx} = \sqrt{a}\cdot\sqrt{a}\,u_{xx} = a\,u_{xx}$.

Our goal is to find a particular solution of (1) and then to construct all the other solutions using property (d). The particular solution we will look for is the one, denoted $Q(x, t)$, which satisfies the *special initial condition*

$$\boxed{Q(x,0) = 1 \quad \text{for } x > 0 \quad Q(x,0) = 0 \quad \text{for } x < 0.} \tag{3}$$

The reason for this choice is that this initial condition does not change under dilation. We'll find Q in three steps.

Step 1 We'll look for $Q(x, t)$ of the special form

$$Q(x,t) = g(p) \quad \text{where } p = \frac{x}{\sqrt{4kt}} \tag{4}$$

and g is a function of only one variable (to be determined). (The $\sqrt{4k}$ factor is included only to simplify a later formula.)

Why do we expect Q to have this special form? Because property (e) says that equation (1) doesn't "see" the dilation $x \to \sqrt{a}\,x$, $t \to at$. Clearly, (3) doesn't change at all under the dilation. So $Q(x, t)$, which is defined by conditions (1) and (3), ought not see the dilation either. How could that happen? In only one way: if Q depends on x and t solely through the combination $x/\sqrt{t}$. For the dilation takes $x/\sqrt{t}$ into $\sqrt{a}x/\sqrt{at} = x/\sqrt{t}$. Thus let $p = x/\sqrt{4kt}$ and look for Q which satisfies (1) and (3) and has the form (4).

Step 2 Using (4), we convert (1) into an ODE for g by use of the chain rule:

$$\begin{aligned}
Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p)\\
Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p)\\
Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p)\\
0 = Q_t - kQ_{xx} &= \frac{1}{t}\left[-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right].
\end{aligned}$$

Thus

$$g'' + 2pg' = 0.$$

This ODE is easily solved using the integrating factor $\exp\int 2p\,dp = \exp(p^2)$. We get $g'(p) = c_1\exp(-p^2)$ and

$$Q(x, t) = g(p) = c_1\int e^{-p^2}dp + c_2.$$

Step 3 We find a completely explicit formula for Q. We've just shown that

$$Q(x, t) = c_1\int_0^{x/\sqrt{4kt}} e^{-p^2}dp + c_2.$$

This formula is valid only for $t > 0$. Now use (3), expressed as a limit as follows.

$$\text{If } x > 0,\quad 1 = \lim_{t\searrow 0} Q = c_1\int_0^{+\infty} e^{-p^2}dp + c_2 = c_1\frac{\sqrt{\pi}}{2} + c_2.$$

$$\text{If } x < 0,\quad 0 = \lim_{t\searrow 0} Q = c_1\int_0^{-\infty} e^{-p^2}dp + c_2 = -c_1\frac{\sqrt{\pi}}{2} + c_2.$$

See Exercise 6. Here $\lim_{t \searrow 0}$ means limit from the right. This determines the coefficients $c_1 = 1/\sqrt{\pi}$ and $c_2 = \frac{1}{2}$. Therefore, Q is the function

$$\boxed{Q(x, t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-p^2}\, dp} \tag{5}$$

for $t > 0$. Notice that it does indeed satisfy (1), (3), and (4).

Step 4 Having found Q, we now *define* $S = \partial Q/\partial x$. (The explicit formula for S will be written below.) By property (b), S is also a solution of (1). Given any function ϕ, we also *define*

$$\boxed{u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)\, dy \quad \text{for } t > 0.} \tag{6}$$

By property (d), u is another solution of (1). *We claim that u is the unique solution of* (1), (2). To verify the validity of (2), we write

$$\begin{aligned} u(x, t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x - y, t)\phi(y)\, dy \\ &= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x - y, t)]\phi(y)\, dy \\ &= +\int_{-\infty}^{\infty} Q(x - y, t)\phi'(y)\, dy - Q(x - y, t)\phi(y)\Big|_{y=-\infty}^{y=+\infty} \end{aligned}$$

upon integrating by parts. We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Therefore,

$$\begin{aligned} u(x, 0) &= \int_{-\infty}^{\infty} Q(x - y, 0)\phi'(y)\, dy \\ &= \int_{-\infty}^{x} \phi'(y)\, dy = \phi\Big|_{-\infty}^{x} = \phi(x) \end{aligned}$$

because of the initial condition for Q and the assumption that $\phi(-\infty) = 0$. This is the initial condition (2). We conclude that (6) is our solution formula, where

$$\boxed{S = \frac{\partial Q}{\partial x} = \frac{1}{2\sqrt{\pi kt}} e^{-x^2/4kt} \quad \text{for } t > 0.} \tag{7}$$

That is,

$$\boxed{u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} \phi(y)\, dy.} \tag{8}$$

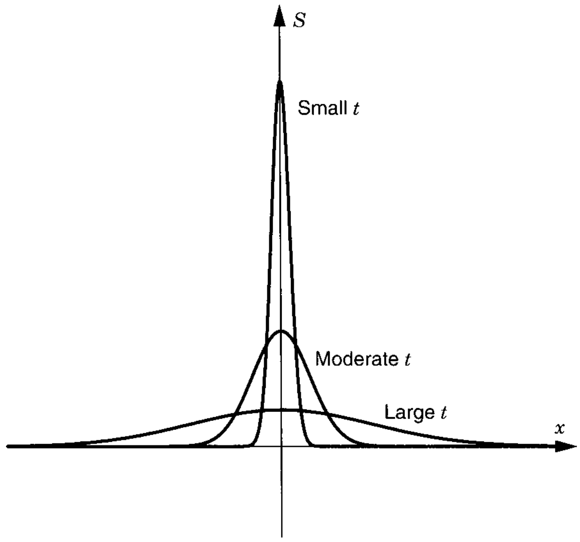

Figure 1

$S(x, t)$ is known as the *source function, Green's function, fundamental solution, gaussian*, or *propagator* of the diffusion equation, or simply the *diffusion kernel*. It gives the solution of (1),(2) with any initial datum ϕ. The formula only gives the solution for $t > 0$. When $t = 0$ it makes no sense. □

The *source function* $S(x, t)$ is defined for all real x and for all $t > 0$. $S(x, t)$ is positive and is even in x $[S(-x, t) = S(x, t)]$. It looks like Figure 1 for various values of t. For large t, it is very spread out. For small t, it is a very tall thin spike (a "*delta function*") of height $(4\pi kt)^{-1/2}$. The area under its graph is

$$\int_{-\infty}^{\infty} S(x, t)\, dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2} dq = 1$$

by substituting $q = x/\sqrt{4kt}$, $dq = (dx)/\sqrt{4kt}$ (see Exercise 7). Now look more carefully at the sketch of $S(x, t)$ for a very small t. If we cut out the tall spike, the rest of $S(x, t)$ is very small. Thus

$$\max_{|x|>\delta} S(x, t) \to 0 \qquad \text{as} \quad t \to 0 \tag{9}$$

Notice that the value of the solution $u(x, t)$ given by (6) is a kind of weighted *average* of the initial values around the point x. Indeed, we can write

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)\, dy \simeq \sum_t S(x - y_i, t)\phi(y_i)\Delta y_i$$

approximately. This is the average of the solutions $S(x - y_i, t)$ with the weights $\phi(y_i)$. For very small t, the source function is a spike so that the formula exaggerates the values of ϕ near x. For *any* $t > 0$ the solution is a spread-out version of the initial values at $t = 0$.

Here's the physical interpretation. Consider diffusion. $S(x - y, t)$ represents the result of a unit mass (say, 1 gram) of substance located at time zero exactly at the position y which is diffusing (spreading out) as time advances. For any initial distribution of concentration, the amount of substance initially in the interval Δy spreads out in time and contributes approximately the term $S(x - y_i, t)\phi(y_i)\Delta y_i$. All these contributions are added up to get the whole distribution of matter. Now consider heat flow. $S(x - y, t)$ represents the result of a "hot spot" at y at time 0. The hot spot is cooling off and spreading its heat along the rod.

Another physical interpretation is brownian motion, where particles move randomly in space. For simplicity, we assume that the motion is one-dimensional; that is, the particles move along a tube. Then the probability that a particle which begins at position x ends up in the interval (a, b) at time t is precisely $\int_a^b S(x - y, t)\, dy$ for some constant k, where S is defined in (7). In other words, if we let $u(x, t)$ be the probability density (probability per unit length) and if the initial probability density is $\phi(x)$, then the probability at all later times is given by formula (6). That is, $u(x, t)$ satisfies the diffusion equation.

It is usually impossible to evaluate integral (8) completely in terms of elementary functions. Answers to particular problems, that is, to particular initial data $\phi(x)$, are sometimes expressible in terms of the *error function* of statistics,

$$\mathscr{E}\mathrm{rf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp. \tag{10}$$

Notice that $\mathscr{E}\mathrm{rf}(0) = 0$. By Exercise 6, $\lim\limits_{x\to+\infty} \mathscr{E}\mathrm{rf}(x) = 1$.

Example 1.

From (5) we can write $Q(x, t)$ in terms of $\mathscr{E}\mathrm{rf}$ as

$$Q(x,\ t) = \frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right). \qquad \square$$

Example 2.

Solve the diffusion equation with the initial condition $u(x, 0) = e^{-x}$. To do so, we simply plug this into the general formula (8):

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} e^{-y}\, dy.$$

This is one of the few fortunate examples that can be integrated. The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

Completing the square in the y variable, it is

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

We let $p = (y + 2kt - x)/\sqrt{4kt}$ so that $dp = dy/\sqrt{4kt}$. Then

$$u(x, t) = e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} \frac{dp}{\sqrt{\pi}} = e^{kt-x}.$$

By the maximum principle, a solution in a bounded interval cannot grow in time. However, this particular solution grows, rather than decays, in time. The reason is that the left side of the rod is initially very hot [$u(x, 0) \to +\infty$ as $x \to -\infty$] and the heat gradually diffuses throughout the rod. □

EXERCISES

1. Solve the diffusion equation with the initial condition

$$\phi(x) = 1 \quad \text{for } |x| < l \qquad \text{and} \qquad \phi(x) = 0 \quad \text{for } |x| > l.$$

Write your answer in terms of $\mathscr{E}\text{rf}(x)$.

2. Do the same for $\phi(x) = 1$ for $x > 0$ and $\phi(x) = 3$ for $x < 0$.
3. Use (8) to solve the diffusion equation if $\phi(x) = e^{3x}$. (You may also use Exercises 6 and 7 below.)
4. Solve the diffusion equation if $\phi(x) = e^{-x}$ for $x > 0$ and $\phi(x) = 0$ for $x < 0$.
5. Prove properties (a) to (e) of the diffusion equation (1).
6. Compute $\int_0^\infty e^{-x^2}\,dx$. (*Hint:* This is a function that *cannot* be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2}\,dx \cdot \int_0^\infty e^{-y^2}\,dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)
7. Use Exercise 6 to show that $\int_{-\infty}^\infty e^{-p^2}\,dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^{\infty} S(x, t)\,dx = 1.$$

8. Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \le |x| < \infty} S(x, t) \to 0 \qquad \text{as } t \to 0.$$

[This means that the tail of $S(x, t)$ is "uniformly small".]

9. Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following special method. First show that u_{xxx} satisfies the diffusion equation with *zero* initial condition. Therefore, by uniqueness, $u_{xxx} \equiv 0$. Integrating this result thrice, obtain $u(x, t) = A(t)x^2 + B(t)x + C(t)$. Finally, it's easy to solve for A, B, and C by plugging into the original problem.
10. (a) Solve Exercise 9 using the general formula discussed in the text. This expresses $u(x, t)$ as a certain integral. Substitute $p = (x - y)/\sqrt{4kt}$ in this integral.
    (b) Since the solution is unique, the resulting formula must agree with the answer to Exercise 9. Deduce the value of
$$\int_{-\infty}^{\infty} p^2 e^{-p^2}\, dp.$$
11. (a) Consider the diffusion equation on the whole line with the usual initial condition $u(x, 0) = \phi(x)$. If $\phi(x)$ is an *odd* function, show that the solution $u(x, t)$ is also an *odd* function of x. (*Hint:* Consider $u(-x, t) + u(x, t)$ and use the uniqueness.)
    (b) Show that the same is true if "odd" is replaced by "even."
    (c) Show that the analogous statements are true for the wave equation.
12. The purpose of this exercise is to calculate $Q(x, t)$ approximately for large t. Recall that $Q(x, t)$ is the temperature of an infinite rod that is initially at temperature 1 for $x > 0$, and 0 for $x < 0$.
    (a) Express $Q(x, t)$ in terms of $\mathscr{E}\mathrm{rf}$.
    (b) Find the Taylor series of $\mathscr{E}\mathrm{rf}(x)$ around $x = 0$. (*Hint:* Expand e^z, substitute $z = -y^2$, and integrate term by term.)
    (c) Use the first two nonzero terms in this Taylor expansion to find an approximate formula for $Q(x, t)$.
    (d) *Why* is this formula a good approximation for x fixed and t large?
13. Prove from first principles that $Q(x, t)$ *must* have the form (4), as follows.
    (a) Assuming uniqueness show that $Q(x, t) = Q(\sqrt{a}\, x, at)$. This identity is valid for all $a > 0$, all $t > 0$, and all x.
    (b) Choose $a = 1/(4kt)$.
14. Let $\phi(x)$ be a continuous function such that $|\phi(x)| \le Ce^{ax^2}$. Show that formula (8) for the solution of the diffusion equation makes sense for $0 < t < 1/(4ak)$, but not necessarily for larger t.
15. Prove the uniqueness of the diffusion problem with Neumann boundary conditions:
$$u_t - ku_{xx} = f(x, t) \quad \text{for } 0 < x < l, t > 0 \quad u(x, 0) = \phi(x)$$
$$u_x(0, t) = g(t) \quad u_x(l, t) = h(t)$$
by the energy method.

16. Solve the diffusion equation with constant dissipation:

$$u_t - ku_{xx} + bu = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* Make the change of variables $u(x,t) = e^{-bt}v(x,t)$.)

17. Solve the diffusion equation with variable dissipation:

$$u_t - ku_{xx} + bt^2u = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where $b > 0$ is a constant. (*Hint:* The solutions of the ODE $w_t + bt^2w = 0$ are $Ce^{-bt^3/3}$. So make the change of variables $u(x,t) = e^{-bt^3/3}v(x,t)$ and derive an equation for v.)

18. Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0 \qquad \text{for } -\infty < x < \infty \qquad \text{with } u(x,0) = \phi(x),$$

where V is a constant. (*Hint:* Go to a moving frame of reference by substituting $y = x - Vt$.)

19. (a) Show that $S_2(x, y, t) = S(x,t)S(y,t)$ satisfies the diffusion equation $S_t = k(S_{xx} + S_{yy})$.
    (b) Deduce that $S_2(x, y, t)$ is the source function for two-dimensional diffusions.

2.5 COMPARISON OF WAVES AND DIFFUSIONS

We have seen that the basic property of waves is that information gets transported in both directions at a finite speed. The basic property of diffusions is that the initial disturbance gets spread out in a smooth fashion and gradually disappears. The fundamental properties of these two equations can be summarized in the following table.

Property		Waves	Diffusions
(i)	Speed of propagation?	Finite ($\leq c$)	Infinite
(ii)	Singularities for $t > 0$?	Transported along characteristics (speed $= c$)	Lost immediately
(iii)	Well-posed for $t > 0$?	Yes	Yes (at least for bounded solutions)
(iv)	Well-posed for $t < 0$?	Yes	No
(v)	Maximum principle	No	Yes
(vi)	Behavior as $t \to +\infty$?	Energy is constant so does not decay	Decays to zero (if ϕ integrable)
(vii)	Information	Transported	Lost gradually

For the wave equation we have seen most of these properties already. That there is no maximum principle is easy to see. Generally speaking, the wave equation just moves information along the characteristic lines. In more than one dimension we'll see that it spreads information in expanding circles or spheres.

For the diffusion equation we discuss property (ii), that singularities are immediately lost, in Section 3.5. The solution is differentiable to all orders even if the initial data are not. Properties (iii), (v), and (vi) have been shown already. The fact that information is gradually lost [property (vii)] is clear from the graph of a typical solution, for instance, from $S(x, t)$.

As for property (i) for the diffusion equation, notice from formula (2.4.8) that the value of $u(x, t)$ depends on the values of the initial datum $\phi(y)$ for *all* y, where $-\infty < y < \infty$. Conversely, the value of ϕ at a point x_0 has an *immediate effect everywhere* (for $t > 0$), even though most of its effect is only for a short time near x_0. Therefore, the *speed of propagation is infinite*. Exercise 2(b) shows that solutions of the diffusion equation can travel at any speed. This is in stark contrast to the wave equation (and all hyperbolic equations).

As for (iv), there are several ways to see that *the diffusion equation is not well-posed for* $t < 0$ ("backward in time"). One way is the following. Let

$$u_n(x, t) = \frac{1}{n} \sin nx \, e^{-n^2 kt}. \tag{1}$$

You can check that this satisfies the diffusion equation for all x, t. Also, $u_n(x, 0) = n^{-1} \sin nx \to 0$ uniformly as $n \to \infty$. But consider any $t < 0$, say $t = -1$. Then $u_n(x, -1) = n^{-1} \sin nx \, e^{+kn^2} \to \pm\infty$ uniformly as $n \to \infty$ except for a few x. Thus u_n is close to the zero solution at time $t = 0$ but not at time $t = -1$. This violates the stability, in the uniform sense at least.

Another way is to let $u(x, t) = S(x, t + 1)$. This is a solution of the diffusion equation $u_t = ku_{xx}$ for $t > -1$, $-\infty < x < \infty$. But $u(0, t) \to \infty$ as $t \searrow -1$, as we saw above. So we cannot solve backwards in time with the perfectly nice-looking initial data $(4\pi k)^{-1} e^{-x^2/4}$.

Besides, any physicist knows that heat flow, brownian motion, and so on, are irreversible processes. Going backward leads to chaos.

EXERCISES

1. Show that there is no maximum principle for the wave equation.
2. Consider a traveling wave $u(x, t) = f(x - at)$ where f is a given function of one variable.
   (a) If it is a solution of the wave equation, show that the speed must be $a = \pm c$ (unless f is a linear function).
   (b) If it is a solution of the diffusion equation, find f and show that the speed a is arbitrary.

3. Let u satisfy the diffusion equation $u_t = \frac{1}{2}u_{xx}$. Let

$$v(x,t) = \frac{1}{\sqrt{t}} e^{x^2/2t} v\left(\frac{x}{t}, \frac{1}{t}\right).$$

Show that v satisfies the "backward" diffusion equation $v_t = -\frac{1}{2}v_{xx}$ for $t > 0$.

4. Here is a direct relationship between the wave and diffusion equations. Let $u(x, t)$ solve the wave equation on the whole line with bounded second derivatives. Let

$$v(x,t) = \frac{c}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-s^2c^2/4kt} u\,(x,s)\,ds.$$

(a) Show that $v(x, t)$ solves the diffusion equation!

(b) Show that $\lim_{t\to 0} v(x,t) = u(x,0)$.

(*Hint:* (a) Write the formula as $v(x,t) = \int_{-\infty}^{\infty} H(s,t)u(x,s)\,ds$, where $H(x, t)$ solves the diffusion equation with constant k/c^2 for $t > 0$. Then differentiate $v(x, t)$ using Section A.3. (b) Use the fact that $H(s, t)$ is essentially the source function of the diffusion equation with the spatial variable s.)