3

REFLECTIONS AND SOURCES

In this chapter we solve the simplest reflection problems, when there is only a single point of reflection at one end of a semi-infinite line. In Chapter 4 we shall begin a systematic study of more complicated reflection problems. In Sections 3.3 and 3.4 we solve problems with sources: that is, the inhomogeneous wave and diffusion equations. Finally, in Section 3.5 we analyze the solution of the diffusion equation more carefully.

3.1 DIFFUSION ON THE HALF-LINE

Let's take the domain to be $D =$ the half-line $(0, \infty)$ and take the *Dirichlet boundary condition* at the single endpoint $x = 0$. So the problem is

$$\boxed{\begin{array}{ll} v_t - kv_{xx} = 0 & \text{in } \{0 < x < \infty, \quad 0 < t < \infty\}, \\ v(x, 0) = \phi(x) & \text{for } t = 0 \\ v(0, t) = 0 & \text{for } x = 0 \end{array}} \tag{1}$$

The PDE is supposed to be satisfied in the open region $\{0 < x < \infty, 0 < t < \infty\}$. If it exists, we know that the solution $v(x, t)$ of this problem is unique because of our discussion in Section 2.3. It can be interpreted, for instance, as the temperature in a very long rod with one end immersed in a reservoir of temperature zero and with insulated sides.

We are looking for a solution formula analogous to (2.4.8). In fact, we shall reduce our new problem to our old one. Our method uses the idea of an *odd function*. Any function $\psi(x)$ that satisfies $\psi(-x) \equiv -\psi(+x)$ is called an odd function. Its graph $y = \psi(x)$ is symmetric with respect to the origin

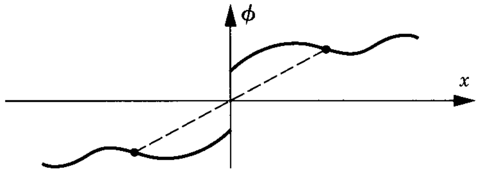

Figure 1

(see Figure 1). Automatically (by putting $x = 0$ in the definition), $\psi(0) = 0$. For a detailed discussion of odd and even functions, see Section 5.2.

Now the initial datum $\phi(x)$ of our problem is defined only for $x \geq 0$. Let ϕ_{odd} be the unique *odd extension* of ϕ to the whole line. That is,

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & \text{for } x > 0 \\ -\phi(-x) & \text{for } x < 0 \\ 0 & \text{for } x = 0. \end{cases} \tag{2}$$

The extension concept too is discussed in Section 5.2.

Let $u(x, t)$ be the solution of

$$\begin{aligned} u_t - ku_{xx} &= 0 \\ u(x, 0) &= \phi_{\text{odd}}(x) \end{aligned} \tag{3}$$

for the *whole line* $-\infty < x < \infty$, $0 < t < \infty$. According to Section 2.3, it is given by the formula

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y,\ t)\phi_{\text{odd}}(y)\,dy. \tag{4}$$

Its "restriction,"

$$v(x, t) = u(x, t) \qquad \text{for } x > 0, \tag{5}$$

will be the unique solution of our new problem (1). There is no difference at all between v and u except that the negative values of x are not considered when discussing v.

Why is $v(x, t)$ the solution of (1)? Notice first that $u(x, t)$ must also be an odd function of x (see Exercise 2.4.11). That is, $u(-x, t) = -u(x, t)$. Putting $x = 0$, it is clear that $u(0, t) = 0$. So the boundary condition $v(0, t) = 0$ is *automatically* satisfied! Furthermore, v solves the PDE as well as the initial condition for $x > 0$, simply because it is equal to u for $x > 0$ and u satisfies the same PDE for all x and the same initial condition for $x > 0$.

The explicit formula for $v(x, t)$ is easily deduced from (4) and (5). From (4) and (2) we have

$$u(x,\ t) = \int_0^{\infty} S(x - y,\ t)\phi(y)\,dy - \int_{-\infty}^{0} S(x - y,\ t)\phi(-y)\,dy.$$

Changing the variable $-y$ to $+y$ in the second integral, we get

$$u(x,t) = \int_0^\infty [S(x-y,\ t) - S(x+y,\ t)]\,\phi(y)\,dy.$$

(Notice the change in the limits of integration.) Hence for $0 < x < \infty$, $0 < t < \infty$, we have

$$\boxed{v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} - e^{-(x+y)^2/4kt}]\,\phi(y)\,dy.} \qquad (6)$$

This is the complete solution formula for (1).

We have just carried out the *method of odd extensions* or *reflection method*, so called because the graph of $\phi_{\text{odd}}(x)$ is the reflection of the graph of $\phi(x)$ across the origin.

Example 1.

Solve (1) with $\phi(x) \equiv 1$. The solution is given by formula (6). This case can be simplified as follows. Let $p = (x-y)/\sqrt{4kt}$ in the first integral and $q = (x+y)/\sqrt{4kt}$ in the second integral. Then

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{x/\sqrt{4kt}} e^{-p^2}\,dp/\sqrt{\pi} - \int_{x/\sqrt{4kt}}^{+\infty} e^{-q^2}\,dq/\sqrt{\pi} \\ &= \left[\frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right] - \left[\frac{1}{2} - \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right)\right] \\ &= \mathscr{E}\mathrm{rf}\left(\frac{x}{\sqrt{4kt}}\right). \end{aligned}$$

□

Now let's play the same game with the *Neumann problem*

$$\boxed{\begin{aligned} &w_t - kw_{xx} = 0 \quad \text{for } 0 < x < \infty,\ 0 < t < \infty \\ &w(x,\ 0) = \phi(x) \\ &w_x(0,\ t) = 0. \end{aligned}} \qquad (7)$$

In this case the reflection method is to use *even*, rather than odd, extensions. An even function is a function ψ such that $\psi(-x) = +\psi(x)$. If ψ is an even function, then differentiation shows that its derivative is an odd function. So automatically its slope at the origin is zero: $\psi'(0) = 0$. If $\phi(x)$ is defined only on the half-line, its *even extension* is defined to be

$$\phi_{\text{even}}(x) = \begin{cases} \phi(x) & \text{for } x \geq 0 \\ +\phi(-x) & \text{for } x \leq 0 \end{cases} \qquad (8)$$

By the same reasoning as we used above, we end up with an explicit formula for $w(x, t)$. It is

$$\boxed{w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}]\, \phi(y)\, dy.} \tag{9}$$

This is carried out in Exercise 3. Notice that the only difference between (6) and (9) is a single minus sign!

Example 2.

Solve (7) with $\phi(x) = 1$. This is the same as Example 1 except for the single sign. So we can copy from that example:

$$u(x,t) = \left[\frac{1}{2} + \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{4kt}\right)\right] + \left[\frac{1}{2} - \frac{1}{2}\mathscr{E}\mathrm{rf}\left(\frac{x}{4kt}\right)\right] = 1.$$

(That was stupid: We could have guessed it!) □

EXERCISES

1. Solve $u_t = ku_{xx}$; $u(x, 0) = e^{-x}$; $u(0, t) = 0$ on the half-line $0 < x < \infty$.
2. Solve $u_t = ku_{xx}$; $u(x, 0) = 0$; $u(0, t) = 1$ on the half-line $0 < x < \infty$.
3. Derive the solution formula for the half-line Neumann problem $w_t - kw_{xx} = 0$ for $0 < x < \infty$, $0 < t < \infty$; $w_x(0, t) = 0$; $w(x, 0) = \phi(x)$.
4. Consider the following problem with a Robin boundary condition:

$$\begin{array}{lll} \text{DE:} & u_t = ku_{xx} & \text{on the half-line } 0 < x < \infty \\ & & \quad \text{(and } 0 < t < \infty) \\ \text{IC:} & u(x,0) = x & \text{for } t = 0 \text{ and } 0 < x < \infty \\ \text{BC:} & u_x(0,t) - 2u(0,t) = 0 & \text{for } x = 0. \end{array} \tag{$*$}$$

The purpose of this exercise is to verify the solution formula for (*). Let $f(x) = x$ for $x > 0$, let $f(x) = x + 1 - e^{2x}$ for $x < 0$, and let

$$v(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\, dy.$$

(a) What PDE and initial condition does $v(x, t)$ satisfy for $-\infty < x < \infty$?
(b) Let $w = v_x - 2v$. What PDE and initial condition does $w(x, t)$ satisfy for $-\infty < x < \infty$?
(c) Show that $f'(x) - 2f(x)$ is an odd function (for $x \neq 0$).
(d) Use Exercise 2.4.11 to show that w is an odd function of x.

(e) Deduce that $v(x, t)$ satisfies (*) for $x > 0$. Assuming uniqueness, deduce that the solution of (*) is given by

$$u(x,\ t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4kt} f(y)\,dy.$$

5. (a) Use the method of Exercise 4 to solve the Robin problem:

$$\begin{array}{rll} \text{DE:} & u_t = ku_{xx} & \text{on the half-line } 0 < x < \infty \\ & & \quad (\text{and } 0 < t < \infty) \\ \text{IC:} & u(x,0) = x & \text{for } t = 0 \text{ and } 0 < x < \infty \\ \text{BC:} & u_x(0,t) - hu(0,t) = 0 & \text{for } x = 0, \end{array}$$

where h is a constant.

(b) Generalize the method to the case of general initial data $\phi(x)$.

3.2 REFLECTIONS OF WAVES

Now we try the same kind of problem for the wave equation as we did in Section 3.1 for the diffusion equation. We again begin with the *Dirichlet problem* on the half-line $(0, \infty)$. Thus the problem is

$$\boxed{\begin{array}{rlll} \text{DE}: & v_{tt} - c^2 v_{xx} = 0 & & \text{for } 0 < x < \infty \\ & & & \text{and } -\infty < t < \infty \\ \text{IC}: & v(x,0) = \phi(x), & v_t(x,0) = \psi(x) & \text{for } t = 0 \\ & & & \text{and } 0 < x < \infty \\ \text{BC}: & v(0,t) = 0 & & \text{for } x = 0 \\ & & & \text{and } -\infty < t < \infty. \end{array}} \qquad (1)$$

The reflection method is carried out in the same way as in Section 3.1. Consider the *odd* extensions of both of the initial functions to the whole line, $\phi_{\text{odd}}(x)$ and $\psi_{\text{odd}}(x)$. Let $u(x, t)$ be the solution of the initial-value problem on $(-\infty, \infty)$ with the initial data ϕ_{odd} and ψ_{odd}. Then $u(x, t)$ is once again an odd function of x (see Exercise 2.1.7). Therefore, $u(0, t) = 0$, so that the boundary condition is satisfied automatically. Define $v(x,t) = u(x,t)$ for $0 < x < \infty$ [the restriction of u to the half-line]. Then $v(x, t)$ is precisely the solution we are looking for. From the formula in Section 2.1, we have for $x \geq 0$,

$$v(x,t) = u(x,\ t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\,dy.$$

Let's "unwind" this formula, recalling the meaning of the odd extensions. First we notice that for $x > c|t|$ only positive arguments occur in the formula,

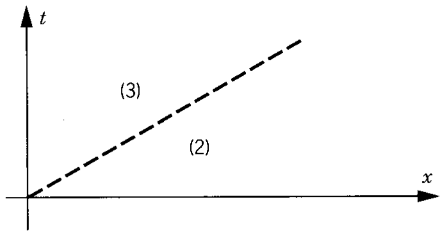

Figure 1

so that $u(x, t)$ is given by the *usual* formula:

$$\boxed{\begin{aligned} v(x,t) &= \frac{1}{2}[\phi(x+ct)+\phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)\,dy \\ &\text{for } x > c|t|. \end{aligned}} \tag{2}$$

But in the *other* region $0 < x < c|t|$, we have $\phi_{\text{odd}}(x-ct) = -\phi(ct-x)$, and so on, so that

$$v(x,t) = \frac{1}{2}[\phi(x+ct)-\phi(ct-x)] + \frac{1}{2c}\int_0^{x+ct}\psi(y)\,dy + \frac{1}{2c}\int_{x-ct}^{0}[-\psi(-y)]\,dy.$$

Notice the switch in signs! In the last term we change variables $y \to -y$ to get $1/2c\int_{ct-x}^{ct+x}\psi(y)\,dy$. Therefore,

$$\boxed{v(x,\ t) = \frac{1}{2}[\phi(ct+x)-\phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x}\psi(y)\,dy} \tag{3}$$

for $0 < x < c|t|$. The complete solution is given by the pair of formulas (2) and (3). The two regions are sketched in Figure 1 for $t > 0$.

Graphically, the result can be interpreted as follows. Draw the backward characteristics from the point (x, t). In case (x, t) is in the region $x < ct$, one of the characteristics hits the t axis ($x = 0$) before it hits the x axis, as indicated in Figure 2. The formula (3) shows that *the reflection induces a change of*

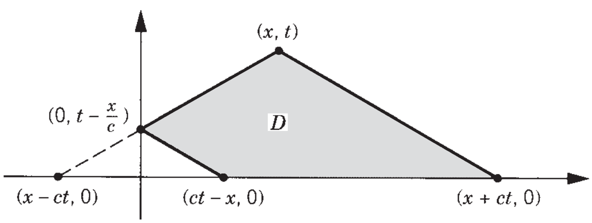

Figure 2

sign. The value of $v(x, t)$ now depends on the values of ϕ at the pair of points $ct \pm x$ and on the values of ψ in the short interval between these points. Note that the other values of ψ have canceled out. The shaded area D in Figure 2 is called the *domain of dependence of the point* (x, t).

The case of the Neumann problem is left as an exercise.

THE FINITE INTERVAL

Now let's consider the guitar string with fixed ends:

$$\boxed{\begin{gathered} v_{tt} = c^2 v_{xx} \quad v(x, 0) = \phi(x) \quad v_t(x, 0) = \psi(x) \quad \text{for } 0 < x < l, \\ v(0, t) = v(l, t) = 0. \end{gathered}} \tag{4}$$

This problem is much more difficult because a typical wave will bounce back and forth an infinite number of times. Nevertheless, let's use the method of reflection. This is a bit tricky, so you are invited to skip the rest of this section if you wish.

The initial data $\phi(x)$ and $\psi(x)$ are now given only for $0 < x < l$. We extend them to the whole line to be "*odd*" with respect to *both* $x = 0$ *and* $x = l$:

$$\phi_{\text{ext}}(-x) = -\phi_{\text{ext}}(x) \quad \text{and} \quad \phi_{\text{ext}}(2l - x) = -\phi_{\text{ext}}(x).$$

The simplest way to do this is to define

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & \text{for} \quad 0 < x < l \\ -\phi(-x) & \text{for} \quad -l < x < 0 \\ \text{extended to be of period } 2l. \end{cases}$$

See Figure 3 for an example. And see Section 5.2 for further discussion. "Period $2l$" means that $\phi_{\text{ext}}(x + 2l) = \phi_{\text{ext}}(x)$ for all x. We do exactly the same for $\psi(x)$ (defined for $0 < x < l$) to get $\psi_{\text{ext}}(x)$ defined for $-\infty < x < \infty$.

Now let $u(x, t)$ be the solution of the infinite line problem with the extended initial data. Let v be the restriction of u to the interval $(0, l)$. Thus $v(x, t)$ is

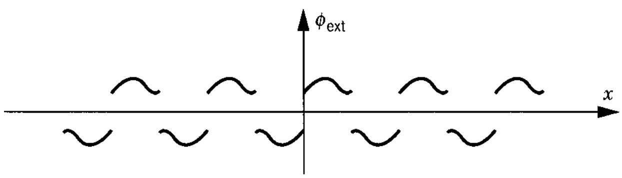

Figure 3

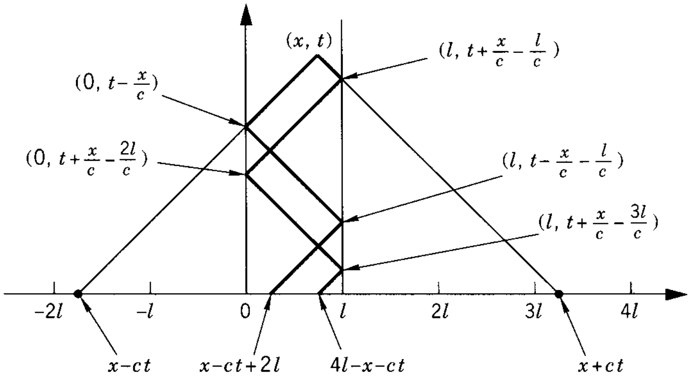

Figure 4

given by the formula

$$v(x,t) = \frac{1}{2}\phi_{\text{ext}}(x+ct) + \frac{1}{2}\phi_{\text{ext}}(x-ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)\,ds \qquad (5)$$

for $0 \le x \le l$. This simple formula contains all the information we need. But to see it explicitly we must unwind the definitions of ϕ_{ext} and ψ_{ext}. This will give a resulting formula which appears quite complicated because it includes a precise description of *all* the reflections of the wave at both of the boundary points $x = 0$ and $x = l$.

The way to understand the explicit result we are about to get is by drawing a space-time diagram (Figure 4). From the point (x, t), we draw the two characteristic lines and reflect them each time they hit the boundary. We keep track of the change of sign at each reflection. We illustrate the result in Figure 4 for the case of a typical point (x, t). We also illustrate in Figure 5 the definition of the extended function $\phi_{\text{ext}}(x)$. (The same picture is valid for ψ_{ext}.) For instance, for the point (x, t) as drawn in Figures 4 and 5, we have

$$\phi_{\text{ext}}(x+ct) = -\phi(4l-x-ct) \quad \text{and} \quad \phi_{\text{ext}}(x-ct) = +\phi(x-ct+2l).$$

The minus coefficient on $-\phi(-x-ct+4l)$ comes from the odd number of reflections ($=3$). The plus coefficient on $\phi(x-ct+2l)$ comes from the even

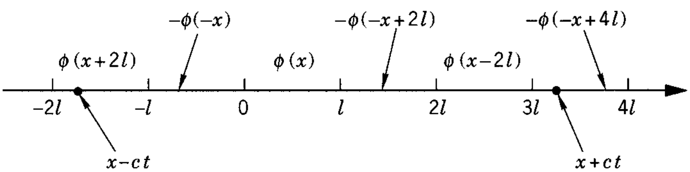

Figure 5

number of reflections (= 2). Therefore, the general formula (5) reduces to

$$
\begin{aligned}
v(x,t) = {} & \frac{1}{2}\phi(x-ct+2l) - \frac{1}{2}\phi(4l-x-ct) \\
& + \frac{1}{2c}\left[\int_{x-ct}^{-l} \psi(y+2l)\,dy + \int_{-l}^{0} -\psi(-y)\,dy \right. \\
& + \int_{0}^{1} \psi(y)\,dy + \int_{l}^{2l} -\psi(-y+2l)\,dy \\
& \left. + \int_{2l}^{3l} \psi(y-2l)\,dy + \int_{3l}^{x+ct} -\psi(-y+4l)\,dy \right]
\end{aligned}
$$

But notice that there is an exact cancellation of the four middle integrals, as we see by changing $y \to -y$ and $y - 2l \to -y + 2l$. So, changing variables in the two remaining integrals, the formula simplifies to

$$
\begin{aligned}
v(x,t) = {} & \frac{1}{2}\phi(x-ct+2l) - \frac{1}{2}\phi(4l-x-ct) \\
& + \frac{1}{2c}\int_{x-ct+2l}^{l} \psi(s)\,ds + \frac{1}{2c}\int_{l}^{4l-x-ct} \psi(s)\,ds.
\end{aligned}
$$

Therefore, we end up with the formula

$$
\boxed{v(x,t) = \frac{1}{2}\phi(x-ct+2l) - \frac{1}{2}\phi(4l-x-ct) + \int_{x-ct+2l}^{4l-x-ct} \psi(s)\frac{ds}{2c}} \qquad (6)
$$

at the point (x, t) *illustrated*, which has three reflections on one end and two on the other. Formula (6) is valid only for such points.

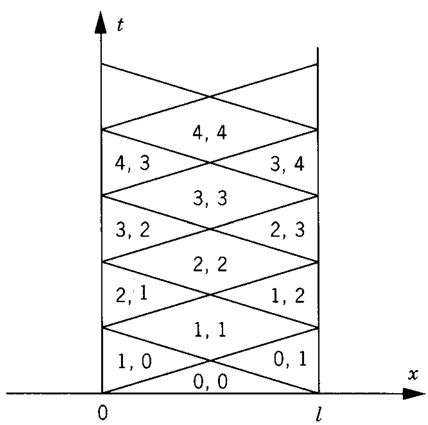

Figure 6

The solution formula at any other point (x, t) is characterized by the number of reflections at each end ($x = 0, l$). This divides the space-time picture into diamond-shaped regions as illustrated in Figure 6. *Within each diamond the solution $v(x, t)$ is given by a different formula.* Further examples may be found in the exercises.

The formulas explain in detail how the solution looks. However, the method is impossible to generalize to two- or three-dimensional problems, nor does it work for the diffusion equation at all. Also, it is very complicated! Therefore, in Chapter 4 we shall introduce a completely different method (Fourier's) for solving problems on a finite interval.

EXERCISES

1. Solve the Neumann problem for the wave equation on the half-line $0 < x < \infty$.
2. The longitudinal vibrations of a semi-infinite flexible rod satisfy the wave equation $u_{tt} = c^2 u_{xx}$ for $x > 0$. Assume that the end $x = 0$ is free ($u_x = 0$); it is initially at rest but has a constant initial velocity V for $a < x < 2a$ and has zero initial velocity elsewhere. Plot u versus x at the times $t = 0, a/c, 3a/2c, 2a/c$, and $3a/c$.
3. A wave $f(x + ct)$ travels along a semi-infinite string ($0 < x < \infty$) for $t < 0$. Find the vibrations $u(x, t)$ of the string for $t > 0$ if the end $x = 0$ is fixed.
4. Repeat Exercise 3 if the end is free.
5. Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0, t) = 0$, $u(x, 0) \equiv 1$, $u_t(x, 0) \equiv 0$ using the reflection method. This solution has a singularity; find its location.
6. Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < \infty$, $0 \le t < \infty$, $u(x, 0) = 0$, $u_t(x, 0) = V$,

$$u_t(0, t) + a u_x(0, t) = 0,$$

where V, a, and c are positive constants and $a > c$.
7. (a) Show that $\phi_{\text{odd}}(x) = (\text{sign } x)\phi(|x|)$.
   (b) Show that $\phi_{\text{ext}}(x) = \phi_{\text{odd}}(x - 2l[x/2l])$, where $[\cdot]$ denotes the greatest integer function.
   (c) Show that

$$\phi_{\text{ext}}(x) = \begin{cases} \phi\left(x - \left[\dfrac{x}{l}\right] l\right) & \text{if } \left[\dfrac{x}{l}\right] \text{ even} \\ -\phi\left(-x - \left[\dfrac{x}{l}\right] l - l\right) & \text{if } \left[\dfrac{x}{l}\right] \text{ odd.} \end{cases}$$

8. For the wave equation in a finite interval $(0, l)$ with Dirichlet conditions, explain the solution formula within each diamond-shaped region.

9. (a) Find $u(\frac{2}{3}, 2)$ if $u_{tt} = u_{xx}$ in $0 < x < 1, u(x, 0) = x^2(1 - x)$, $u_t(x, 0) = (1 - x)^2, u(0, t) = u(1, t) = 0$.
   (b) Find $u(\frac{1}{4}, \frac{7}{2})$.
10. Solve $u_{tt} = 9u_{xx}$ in $0 < x < \pi/2, u(x, 0) = \cos x, u_t(x, 0) = 0$, $u_x(0, t) = 0, u(\pi/2, t) = 0$.
11. Solve $u_{tt} = c^2 u_{xx}$ in $0 < x < l, u(x, 0) = 0, u_t(x, 0) = x, u(0, t) = u(l, t) = 0$.

3.3 DIFFUSION WITH A SOURCE

In this section we solve the *inhomogeneous* diffusion equation on the whole line,

$$\boxed{\begin{aligned} &u_t - ku_{xx} = f(x, t) \qquad (-\infty < x < \infty, \quad 0 < t < \infty) \\ &u(x, 0) = \phi(x) \end{aligned}} \tag{1}$$

with $f(x, t)$ and $\phi(x)$ arbitrary given functions. For instance, if $u(x, t)$ represents the temperature of a rod, then $\phi(x)$ is the initial temperature distribution and $f(x, t)$ is a source (or sink) of heat provided to the rod at later times.

We will show that the solution of (1) is

$$\boxed{\begin{aligned} u(x, t) = &\int_{-\infty}^{\infty} S(x - y, t)\phi(y)\, dy \\ &+ \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s)\, dy\, ds. \end{aligned}} \tag{2}$$

Notice that there is the usual term involving the initial data ϕ and another term involving the source f. Both terms involve the source function S.

Let's begin by explaining where (2) comes from. Later we will actually prove the validity of the formula. (If a strictly mathematical proof is satisfactory to you, this paragraph and the next two can be skipped.) Our explanation is an analogy. The simplest analogy is the ODE

$$\frac{du}{dt} + Au(t) = f(t), \qquad u(0) = \phi, \tag{3}$$

where A is a constant. Using the integrating factor e^{tA}, the solution is

$$u(t) = e^{-tA}\phi + \int_0^t e^{(s-t)A} f(s)\, ds. \tag{4}$$

A more elaborate analogy is the following. Let's suppose that ϕ is an n-vector, $u(t)$ is an n-vector function of time, and A is a fixed $n \times n$ matrix.

Then (3) is a coupled system of n linear ODEs. In case $f(t) \equiv 0$, the solution of (3) is given as $u(t) = S(t)\phi$, where $S(t)$ is the matrix $S(t) = e^{-tA}$. So in case $f(t) \neq 0$, an integrating factor for (3) is $S(-t) = e^{tA}$. Now we multiply (3) on the left by this integrating factor to get

$$\frac{d}{dt}[S(-t)u(t)] = S(-t)\frac{du}{dt} + S(-t)Au(t) = S(-t)f(t).$$

Integrating from 0 to t, we get

$$S(-t)u(t) - \phi = \int_0^t S(-s)f(s)\,ds.$$

Multiplying this by $S(t)$, we end up with the solution formula

$$u(t) = S(t)\phi + \int_0^t S(t-s)f(s)\,ds. \tag{5}$$

The first term in (5) represents the solution of the homogeneous equation, the second the effect of the source $f(t)$. For a single equation, of course, (5) reduces to (4). □

Now let's return to the original diffusion problem (1). There is an analogy between (2) and (5) which we now explain. The solution of (1) will have two terms. The first one will be the solution of the homogeneous problem, already solved in Section 2.4, namely

$$\int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy = (\mathcal{S}(t)\phi)(x). \tag{6}$$

$S(x - y, t)$ is the source function given by the formula (2.4.7). Here we are using $\mathcal{S}(t)$ to denote the *source operator*, which transforms any function ϕ to the new function given by the integral in (6). (Remember: Operators transform functions into functions.) We can now *guess* what the whole solution to (1) must be. In analogy to formula (5), we guess that the solution of (1) is

$$u(t) = \mathcal{S}(t)\phi + \int_0^t \mathcal{S}(t-s)f(s)\,ds. \tag{7}$$

Formula (7) is exactly the same as (2):

$$\begin{aligned} u(x,t) = &\int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy \\ &+ \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)\,dy\,ds. \end{aligned} \tag{2}$$

The method we have just used to find formula (2) is the operator method.

Proof of (2). All we have to do is verify that the function $u(x, t)$, which is *defined* by (2), in fact satisfies the PDE and IC (1). Since the solution of

(1) is unique, we would then know that $u(x, t)$ is that unique solution. For simplicity, we may as well let $\phi \equiv 0$, since we understand the ϕ term already.

We first verify the PDE. Differentiating (2), assuming $\phi \equiv 0$ and using the rule for differentiating integrals in Section A.3, we have

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y, s)\, dy\, ds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{\partial S}{\partial t}(x-y, t-s) f(y, s)\, dy\, ds \\ &\quad + \lim_{s \to t} \int_{-\infty}^{\infty} S(x-y, t-s) f(y, s)\, dy,\end{aligned}$$

taking special care due to the singularity of $S(x - y, t - s)$ at $t - s = 0$. Using the fact that $S(x - y, t - s)$ satisfies the diffusion equation, we get

$$\begin{aligned}\frac{\partial u}{\partial t} &= \int_0^t \int_{-\infty}^{\infty} k\frac{\partial^2 S}{\partial x^2}(x-y,\ t-s) f(y,\ s)\, dy\, ds \\ &\quad + \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} S(x-y, \epsilon) f(y, t)\, dy.\end{aligned}$$

Pulling the spatial derivative outside the integral and using the initial condition satisfied by S, we get

$$\begin{aligned}\frac{\partial u}{\partial t} &= k\frac{\partial^2}{\partial x^2}\int_0^t \int_{-\infty}^{\infty} S(x-y,\ t-s) f(y,\ s)\, dy\, ds + f(x, t) \\ &= k\frac{\partial^2 u}{\partial x^2} + f(x, t).\end{aligned}$$

This identity is exactly the PDE (1). Second, we verify the initial condition. Letting $t \to 0$, the first term in (2) tends to $\phi(x)$ because of the initial condition of S. The second term is an integral from 0 to 0. Therefore,

$$\lim_{t \to 0} u(x, t) = \phi(x) + \int_0^0 \cdots = \phi(x).$$

This proves that (2) is the unique solution. $\square$

Remembering that $S(x, t)$ is the gaussian distribution (2.4.7), the formula (2) takes the explicit form

$$\begin{aligned}u(x, t) &= \int_0^t \int_{-\infty}^{\infty} S(x-y, t-s) f(y, s)\, dy\, ds \\ &= \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi k(t-s)}} e^{-(x-y)^2/4k(t-s)} f(y, s)\, dy\, ds. \qquad (8)\end{aligned}$$

in the case that $\phi \equiv 0$.

SOURCE ON A HALF-LINE

For inhomogeneous diffusion on the half-line we can use the method of reflection just as in Section 3.1 (see Exercise 1).

Now consider the more complicated problem of a *boundary source* $h(t)$ on the half-line; that is,

$$\begin{aligned} v_t - kv_{xx} &= f(x,t) \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty \\ \boldsymbol{v(0,t)} &= \boldsymbol{h(t)} \\ v(x,0) &= \phi(x). \end{aligned} \tag{9}$$

We may use the following subtraction device to reduce (9) to a simpler problem. Let $V(x,t) = v(x,t) - h(t)$. Then $V(x, t)$ will satisfy

$$\begin{aligned} V_t - kV_{xx} &= f(x,t) - h'(t) \qquad \text{for } 0 < x < \infty, \qquad 0 < t < \infty \\ V(0,t) &= 0 \\ V(x,0) &= \phi(x) - h(0). \end{aligned} \tag{10}$$

To verify (10), just subtract! This new problem has a homogeneous boundary condition to which we can apply the method of reflection. Once we find V, we recover v by $v(x,t) = V(x,t) + h(t)$. This simple subtraction device is often used to reduce one linear problem to another.

The domain of independent variables (x, t) in this case is a quarter-plane with specified conditions on both of its half-lines. If they do not agree at the corner [i.e., if $\phi(0) \neq h(0)$], then the solution is discontinuous there (but continuous everywhere else). This is physically sensible. Think for instance, of suddenly at $t = 0$ sticking a hot iron bar into a cold bath.

For the inhomogeneous *Neumann* problem on the half-line,

$$\begin{aligned} w_t - kw_{xx} &= f(x,t) \qquad \text{for } 0 < x < \infty, \qquad 0 < t < \infty \\ \boldsymbol{w_x(0,t)} &= \boldsymbol{h(t)} \\ w(x,0) &= \phi(x), \end{aligned} \tag{11}$$

we would subtract off the function $xh(t)$. That is, $W(x, t) = w(x, t) - xh(t)$. Differentiation implies that $W_x(0, t) = 0$. Some of these problems are worked out in the exercises.

EXERCISES

1. Solve the inhomogeneous diffusion equation on the half-line with Dirichlet boundary condition:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \qquad (0 < x < \infty, \quad 0 < t < \infty) \\ u(0,t) &= 0 \qquad u(x,0) = \phi(x) \end{aligned}$$

using the method of reflection.

2. Solve the completely inhomogeneous diffusion problem on the half-line

$$v_t - kv_{xx} = f(x,t) \qquad \text{for } 0 < x < \infty, \quad 0 < t < \infty$$
$$v(0,t) = h(t) \qquad v(x,0) = \phi(x),$$

by carrying out the subtraction method begun in the text.

3. Solve the inhomogeneous Neumann diffusion problem on the half-line

$$w_t - kw_{xx} = 0 \qquad \text{for } 0 < x < \infty, \quad 0 < t < \infty$$
$$w_x(0,t) = h(t) \qquad w(x,0) = \phi(x),$$

by the subtraction method indicated in the text.

3.4 WAVES WITH A SOURCE

The purpose of this section is to solve

$$\boxed{u_{tt} - c^2 u_{xx} = f(x,t)} \tag{1}$$

on the whole line, together with the usual initial conditions

$$\boxed{\begin{aligned} u(x,0) &= \phi(x) \\ u_t(x,0) &= \psi(x) \end{aligned}} \tag{2}$$

where $f(x, t)$ is a given function. For instance, $f(x, t)$ could be interpreted as an external force acting on an infinitely long vibrating string.

Because $L = \partial_t^2 - c^2\partial_x^2$ is a linear operator, the solution will be the *sum of three terms*, one for ϕ, one for ψ, and one for f. The first two terms are given already in Section 2.1 and we must find the third term. We'll derive the following formula.

Theorem 1. The unique solution of (1),(2) is

$$\boxed{u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi + \frac{1}{2c}\iint_{\Delta} f} \tag{3}$$

where Δ is the characteristic triangle (see Figure 1).

The double integral in (3) is equal to the iterated integral

$$\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

We will give three different derivations of this formula! But first, let's note what the formula says. It says that the effect of a force f on $u(x, t)$ is obtained

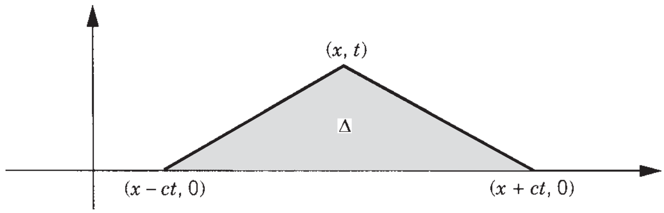

Figure 1

by simply integrating f over the past history of the point (x, t) back to the initial time $t = 0$. This is yet another example of the causality principle.

WELL-POSEDNESS

We first show that the problem (1),(2) is well-posed in the sense of Section 1.5. The well-posedness has three ingredients, as follows. Existence is clear, given that the formula (3) itself is an explicit solution. If ϕ has a continuous second derivative, ψ has a continuous first derivative, and f is continuous, then the formula (3) yields a function u with continuous second partials that satisfies the equation. Uniqueness means that there are no other solutions of (1),(2). This will follow from any one of the derivations given below.

Third, we claim that the problem (1),(2) is stable in the sense of Section 1.5. This means that if the data (ϕ, ψ, f) change a little, then u also changes only a little. To make this precise, we need a way to measure the "nearness" of functions, that is, a *metric* or *norm* on function spaces. We will illustrate this concept using the *uniform norms:*

$$\|w\| = \max_{-\infty<x<\infty} |w(x)|$$

and

$$\|w\|_T = \max_{-\infty<x<\infty,\ 0\le t\le T} |w(x,t)|.$$

Here T is fixed. Suppose that $u_1(x,t)$ is the solution with data $(\phi_1(x), \psi_1(x), f_1(x,t))$ and $u_2(x,t)$ is the solution with data $(\phi_2(x), \psi_2(x), f_2(x,t))$ (six given functions). We have the same formula (3) satisfied by u_1 and by u_2 except for the different data. We subtract the two formulas. We let $u = u_1 - u_2$. Since the area of Δ equals ct^2, we have from (3) the inequality

$$\begin{aligned}|u(x,t)| &\le \max|\phi| + \frac{1}{2c}\cdot\max|\psi|\cdot 2ct + \frac{1}{2c}\cdot\max|f|\cdot ct^2\\ &= \max|\phi| + t\cdot\max|\psi| + \frac{t^2}{2}\cdot\max|f|.\end{aligned}$$

Therefore,

$$\boxed{\|u_1 - u_2\|_T \le \|\phi_1 - \phi_2\| + T\,\|\psi_1 - \psi_2\| + \frac{T^2}{2}\|f_1 - f_2\|_T.} \tag{4}$$

So if $\|\phi_1 - \phi_2\| < \delta$, $\|\psi_1 - \psi_2\| < \delta$, and $\|f_1 - f_2\|_T < \delta$, where δ is small, then

$$\|u_1 - u_2\|_T < \delta(1 + T + T^2) \le \epsilon$$

provided that $\delta \le \epsilon/(1 + T + T^2)$. Since ϵ is arbitrarily small, this argument proves the well-posedness of the problem (1),(2) with respect to the uniform norm.

PROOF OF THEOREM 1

Method of Characteristic Coordinates We introduce the usual characteristic coordinates $\xi = x + ct$, $\eta = x - ct$, (see Figure 2). As in Section 2.1, we have

$$Lu \equiv u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} = f\left(\frac{\xi + \eta}{2}, \frac{\xi - \eta}{2c}\right).$$

We integrate this equation with respect to η, leaving ξ as a constant. Thus $u_\xi = -(1/4c^2)\int^\eta f\,d\eta$. Then we integrate with respect to ξ to get

$$u = -\frac{1}{4c^2}\int^\xi \int^\eta f\,d\eta\,d\xi \tag{5}$$

The lower limits of integration here are arbitrary: They correspond to constants of integration. The calculation is much easier to understand if we fix a point P_0 with coordinates x_0, t_0 and

$$\xi_0 = x_0 + ct_0 \qquad \eta_0 = x_0 - ct_0.$$

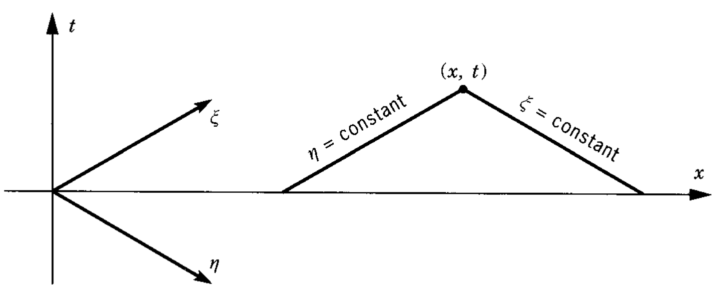

Figure 2

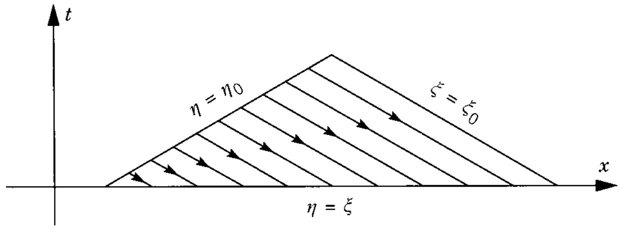

Figure 3

We evaluate (5) at P_0 *and make a particular choice of the lower limits*. Thus

$$\begin{aligned} u(P_0) &= -\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\xi}^{\eta_0} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right) d\eta\, d\xi \\ &= +\frac{1}{4c^2}\int_{\eta_0}^{\xi_0}\int_{\eta_0}^{\xi} f\left(\frac{\xi+\eta}{2}, \frac{\xi-\eta}{2c}\right) d\eta\, d\xi \end{aligned} \tag{6}$$

is a particular solution. As Figure 3 indicates, η now represents a variable going along a line segment to the base $\eta = \xi$ of the triangle Δ from the left-hand edge $\eta = \eta_0$, while ξ runs from the left-hand corner to the right-hand edge. Thus we have integrated over the whole triangle Δ.

The iterated integral, however, is not exactly the double integral over Δ because the coordinate axes are not orthogonal. The original axes (x and t) are orthogonal, so we make a change of variables back to x and t. This amounts to substituting back

$$x = \frac{\xi+\eta}{2} \qquad t = \frac{\xi-\eta}{2c}. \tag{7}$$

A little square in Figure 4 goes into a parallelogram in Figure 5. The change in its area is measured by the jacobian determinant J (see Section A.1). Since

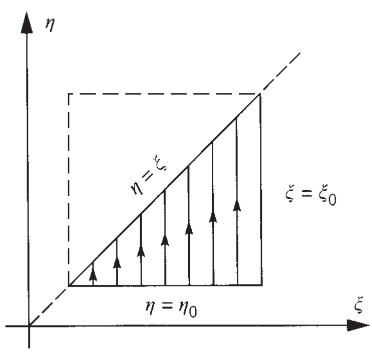

Figure 4

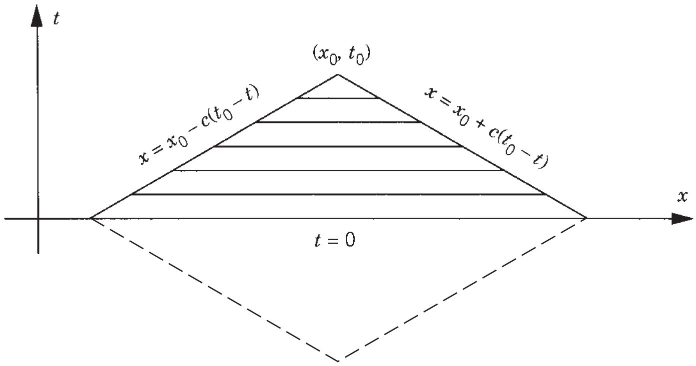

Figure 5

our change of variable is a linear transformation, the jacobian is just the determinant of its coefficient matrix:

$$J = \left|\det\begin{pmatrix} \dfrac{\partial \xi}{\partial x} & \dfrac{\partial \xi}{\partial t} \\ \dfrac{\partial \eta}{\partial x} & \dfrac{\partial \eta}{\partial t} \end{pmatrix}\right| = \left|\det\begin{pmatrix} 1 & c \\ 1 & -c \end{pmatrix}\right| = 2c.$$

Thus $d\eta\, d\xi = J\, dx\, dt = 2c\, dx\, dt$. Therefore, the rule for changing variables in a multiple integral (the jacobian theorem) then gives

$$u(P_0) = \frac{1}{4c^2} \iint_\Delta f(x,t) J\, dx\, dt. \tag{8}$$

This is precisely Theorem 1. The formula can also be written as the iterated integral in x and t:

$$u(x_0, t_0) = \frac{1}{2c} \int_0^{t_0} \int_{x_0 - c(t_0 - t)}^{x_0 + c(t_0 - t)} f(x,t)\, dx\, dt, \tag{9}$$

integrating first over the horizontal line segments in Figure 5 and then vertically.

A variant of the method of characteristic coordinates is to write (1) as the system of two equations

$$u_t + cu_x = v \quad v_t - cv_x = f,$$

the first equation being the definition of v, as in Section 2.1. If we first solve the second equation, then v is a line integral of f over a characteristic line segment $x + ct =$ constant. The first equation then gives $u(x, t)$ by sweeping out these line segments over the characteristic triangle Δ. To carry out this variant is a little tricky, however, and we leave it as an exercise.

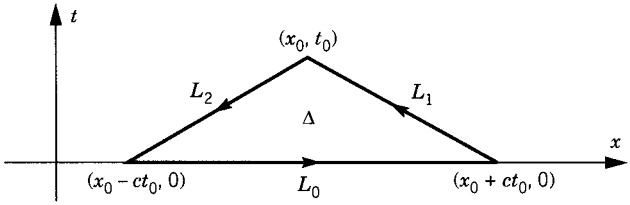

Figure 6

Method Using Green's Theorem In this method we integrate f over the past history triangle Δ. Thus

$$\iint_\Delta f\,dx\,dt = \iint_\Delta (u_{tt} - c^2 u_{xx})\,dx\,dt. \tag{10}$$

But Green's theorem says that

$$\iint_\Delta (P_x - Q_t)\,dx\,dt = \int_{\text{bdy}} P\,dt + Q\,dx$$

for any functions P and Q, where the line integral on the boundary is taken counterclockwise (see Section A.3). Thus we get

$$\iint_\Delta f\,dx\,dt = \int_{L_0+L_1+L_2} (-c^2 u_x dt - u_t\,dx). \tag{11}$$

This is the sum of three line integrals over straight line segments (see Figure 6). We evaluate each piece separately. On L_0, $dt = 0$ and $u_t(x, 0) = \psi(x)$, so that

$$\int_{L_0} = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx.$$

On L_1, $x + ct = x_0 + ct_0$, so that $dx + c\,dt = 0$, whence $-c^2 u_x\,dt - u_t\,dx = cu_x\,dx + cu_t\,dt = c\,du$. (We're in luck!) Thus

$$\int_{L_1} = c\int_{L_1} du = cu(x_0, t_0) - c\phi(x_0 + ct_0).$$

In the same way,

$$\int_{L_2} = -c\int_{L_2} du = -c\phi(x_0 - ct_0) + cu(x_0, t_0).$$

Adding these three results, we get

$$\iint_\Delta f\,dx\,dt = 2cu(x_0, t_0) - c[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx.$$

Thus

$$u(x_0, t_0) = \frac{1}{2c} \iint_\Delta f\, dx\, dt + \frac{1}{2}[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] \\ + \frac{1}{2c} \int_{x_0 - ct_0}^{x_0 + ct_0} \psi(x)\, dx, \tag{12}$$

which is the same as before.

Operator Method This is how we solved the diffusion equation with a source. Let's try it out on the wave equation. The ODE analog is the equation,

$$\frac{d^2u}{dt^2} + A^2 u(t) = f(t), \quad u(0) = \phi, \quad \frac{du}{dt}(0) = \psi. \tag{13}$$

We could think of A^2 as a positive constant (or even a positive square matrix.) The solution of (13) is

$$u(t) = S'(t)\phi + S(t)\psi + \int_0^t S(t-s) f(s)\, ds, \tag{14}$$

where

$$S(t) = A^{-1} \sin tA \quad \text{and} \quad S'(t) = \cos tA. \tag{15}$$

The key to understanding formula (14) is that $S(t)\psi$ is the solution of problem (13) in the case that $\phi = 0$ and $f = 0$.

Let's return to the PDE

$$u_{tt} - c^2 u_{xx} = f(x, t) \quad u(x, 0) = \phi(x) \quad u_t(x, 0) = \psi(x). \tag{16}$$

The basic operator ought to be given by the ψ term. That is,

$$\mathcal{S}(t)\psi = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y)\, dy = v(x, t), \tag{17}$$

where $v(x, t)$ solves $v_{tt} - c^2 v_{xx} = 0$, $v(x, 0) = 0$, $v_t(x, 0) = \psi(x)$. $\mathcal{S}(t)$ is the *source operator*. By (14) we would expect the ϕ term to be $(\partial/\partial t)\mathcal{S}(t)\phi$. In fact,

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{S}(t)\phi &= \frac{\partial}{\partial t} \frac{1}{2c} \int_{x-ct}^{x+ct} \phi(y)\, dy \\ &= \frac{1}{2c} \left[c\phi(x+ct) - (-c)\phi(x-ct) \right], \end{aligned}$$

in agreement with our old formula (2.1.8)! So we must be on the right track.

Let's now take the f term; that is, $\phi = \psi = 0$. By analogy with the last term in (14), the solution *ought* to be

$$u(t) = \int_0^t \mathcal{S}(t-s) f(s)\, ds.$$

That is, using (17),

$$u(x,t) = \int_0^t \left[\frac{1}{2c} \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy \right] ds = \frac{1}{2c} \iint_\Delta f\,dx\,dt.$$

This is once again the same result!

The moral of the operator method is that if you can solve the homogeneous equation, you can also solve the inhomogeneous equation. This is sometimes known as *Duhamel's principle.*

SOURCE ON A HALF-LINE

The solution of the general inhomogeneous problem on a half-line

$$\begin{aligned} \text{DE:} \quad & v_{tt} - c^2 v_{xx} = f(x,t) \quad \text{in} \quad 0 < x < \infty \\ \text{IC:} \quad & v(x,0) = \phi(x) \quad v_t(x,0) = \psi(x) \\ \text{BC:} \quad & v(0,t) = h(t) \end{aligned} \tag{18}$$

is the sum of four terms, one for each data function ϕ, ψ, f, and h. For $x > ct > 0$, the solution has precisely the same form as in (3), with the backward triangle Δ as the domain of dependence. For $0 < x < ct$, however, it is given by

$$v(x,t) = \phi \text{ term} + \psi \text{ term} + \boldsymbol{h}\left(\boldsymbol{t} - \frac{\boldsymbol{x}}{\boldsymbol{c}}\right) + \frac{1}{2c} \iint_D f \tag{19}$$

where $t - x/c$ is the reflection point and D is the shaded region in Figure 3.2.2. The only caveat is that the given conditions had better coincide at the origin. That is, we require that $\phi(0) = h(0)$ and $\psi(0) = h'(0)$. If this were not assumed, there would be a singularity on the characteristic line emanating from the corner.

Let's derive the boundary term $h(t - x/c)$ for $x < ct$. To accomplish this, it is convenient to assume that $\phi = \psi = f = 0$. We shall derive the solution from scratch using the fact that $v(x,t)$ must take the form $v(x,t) = j(x+ct) + g(x-ct)$. From the initial conditions ($\phi = \psi = 0$), we find that $j(s) = g(s) = 0$ for $s > 0$. From the boundary condition we have $h(t) = v(0,t) = g(-ct)$ for $t > 0$. Thus $g(s) = h(-s/c)$ for $s < 0$. Therefore, if $x < ct$, $t > 0$, we have $v(x,\ t) = 0 + h(-[x-ct]/c) = h(t - x/c)$.

FINITE INTERVAL

For a finite interval $(0, l)$ with inhomogeneous boundary conditions $v(0,t) = h(t)$, $v(l,t) = k(t)$, we get the whole series of terms

$$\begin{aligned} v(x,t) = h\left(t - \frac{x}{c}\right) - h\left(t + \frac{x-2l}{c}\right) + h\left(t - \frac{x+2l}{c}\right) + \cdots \\ + k\left(t + \frac{x-l}{c}\right) - k\left(t - \frac{x+l}{c}\right) + k\left(t + \frac{x-3l}{c}\right) + \cdots \end{aligned}$$

(see Exercise 15 and Figure 3.2.4).

EXERCISES

1. Solve $u_{tt} = c^2u_{xx} + xt, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0.$
2. Solve $u_{tt} = c^2u_{xx} + e^{ax}, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0.$
3. Solve $u_{tt} = c^2u_{xx} + \cos x, \quad u(x, 0) = \sin x, \quad u_t(x, 0) = 1 + x.$
4. Show that the solution of the inhomogeneous wave equation

$$u_{tt} = c^2u_{xx} + f, \quad u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x),$$

is the sum of three terms, one each for f, ϕ, and ψ.

5. Let $f(x, t)$ be any function and let $u(x, t) = (1/2c)\iint_\Delta f$, where Δ is the triangle of dependence. Verify directly by differentiation that

$$u_{tt} = c^2u_{xx} + f \quad \text{and} \quad u(x, 0) \equiv u_t(x, 0) \equiv 0.$$

(*Hint:* Begin by writing the formula as the *iterated* integral

$$u(x, t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y, s)\,dy\,ds$$

and differentiate with care using the rule in the Appendix. This exercise is not easy.)

6. Derive the formula for the inhomogeneous wave equation in yet another way.
   (a) Write it as the system

$$u_t + cu_x = v, \quad v_t - cv_x = f.$$

   (b) Solve the first equation for u in terms of v as

$$u(x, t) = \int_0^t v(x - ct + cs, s)\,ds.$$

   (c) Similarly, solve the second equation for v in terms of f.
   (d) Substitute part (c) into part (b) and write as an iterated integral.

7. Let A be a positive-definite $n \times n$ matrix. Let

$$S(t) = \sum_{m=0}^{\infty} \frac{(-1)^m A^{2m} t^{2m+1}}{(2m + 1)!}.$$

   (a) Show that this series of matrices converges uniformly for bounded t and its sum $S(t)$ solves the problem $S''(t) + A^2S(t) = 0,\ S(0) = 0,\ S'(0) = I$, where I is the identity matrix. Therefore, it makes sense to denote $S(t)$ as $A^{-1} \sin tA$ and to denote its derivative $S'(t)$ as $\cos(tA)$.
   (b) Show that the solution of (13) is (14).

8. Show that the source operator for the wave equation solves the problem

$$\mathcal{S}_{tt} - c^2\mathcal{S}_{xx} = 0, \quad \mathcal{S}(0) = 0, \quad \mathcal{S}_t(0) = I,$$

where I is the identity operator.

9. Let $u(t) = \int_0^t \mathcal{S}(t-s) f(s)\,ds$. Using *only* Exercise 8, show that u solves the inhomogeneous wave equation with zero initial data.
10. Use any method to show that $u = 1/(2c) \iint_D f$ solves the inhomogeneous wave equation on the half-line with zero initial and boundary data, where D is the domain of dependence for the half-line.
11. Show by direct substitution that $u(x, t) = h(t - x/c)$ for $x < ct$ and $u(x, t) = 0$ for $x \geq ct$ solves the wave equation on the half-line $(0, \infty)$ with zero initial data and boundary condition $u(0, t) = h(t)$.
12. Derive the solution of the fully inhomogeneous wave equation on the half-line
$$v_{tt} - c^2 v_{xx} = f(x, t) \quad \text{in } 0 < x < \infty$$
$$v(x, 0) = \phi(x), \qquad v_t(x, 0) = \psi(x)$$
$$v(0, t) = h(t),$$
by means of the method using Green's theorem. (*Hint:* Integrate over the domain of dependence.)
13. Solve $u_{tt} = c^2 u_{xx}$ for $0 < x < \infty$,
$u(0, t) = t^2, \quad u(x, 0) = x, \quad u_t(x, 0) = 0.$
14. Solve the homogeneous wave equation on the half-line $(0, \infty)$ with zero initial data and with the Neumann boundary condition $u_x(0, t) = k(t)$. Use any method you wish.
15. Derive the solution of the wave equation in a finite interval with inhomogeneous boundary conditions $v(0, t) = h(t)$, $v(l, t) = k(t)$, and with $\phi = \psi = f = 0$.

3.5 DIFFUSION REVISITED

In this section we make a careful mathematical analysis of the solution of the diffusion equation that we found in Section 2.4. (On the other hand, the formula for the solution of the wave equation is so much simpler that it doesn't require a special justification.)

The solution formula for the diffusion equation is an example of a *convolution*, the convolution of ϕ with S (at a fixed t). It is

$$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\,\phi(y)\,dy = \int_{-\infty}^{\infty} S(z, t)\,\phi(x - z)\,dz, \qquad (1)$$

where $S(z, t) = 1/\sqrt{4\pi kt}\, e^{-z^2/4kt}$. If we introduce the variable $p = z/\sqrt{kt}$, it takes the equivalent form

$$\boxed{u(x, t) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} \phi(x - p\sqrt{kt})\,dp.} \qquad (2)$$

Now we are prepared to state a precise theorem.

Theorem 1. Let $\phi(x)$ be a bounded continuous function for $-\infty < x < \infty$. Then the formula (2) defines an infinitely differentiable function $u(x, t)$ for $-\infty < x < \infty$, $0 < t < \infty$, which satisfies the equation $u_t = ku_{xx}$ and $\lim_{t \searrow 0} u(x, t) = \phi(x)$ for each x.

Proof. The integral converges easily because

$$|u(x,t)| \le \frac{1}{\sqrt{4\pi}} (\max|\phi|) \int_{-\infty}^{\infty} e^{-p^2/4}\, dp = \max|\phi|.$$

(This inequality is related to the maximum principle.) Thus the integral converges uniformly and absolutely. Let us show that $\partial u/\partial x$ exists. It equals $\int (\partial S/\partial x)(x-y,t)\phi(y)\,dy$ provided that this new integral also converges absolutely. Now

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\partial S}{\partial x}(x-y,t)\phi(y)\,dy &= -\frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \frac{x-y}{2kt} e^{-(x-y)^2/4kt} \phi(y)\,dy \\
&= \frac{c}{\sqrt{t}} \int_{-\infty}^{\infty} p e^{-p^2/4} \phi(x - p\sqrt{kt})\,dp \\
&\le \frac{c}{\sqrt{t}} (\max|\phi|) \int_{-\infty}^{\infty} |p|\, e^{-p^2/4}\, dp,
\end{aligned}$$

where c is a constant. The last integral is finite. So this integral also converges uniformly and absolutely. Therefore, $u_x = \partial u/\partial x$ exists and is given by this formula. All derivatives of all orders ($u_t, u_{xt}, u_{xx}, u_{tt}, \ldots$) work the same way because each differentiation brings down a power of p so that we end up with convergent integrals like $\int p^n e^{-p^2/4}\,dp$. So $u(x, t)$ is differentiable to all orders. Since $S(x, t)$ satisfies the diffusion equation for $t > 0$, so does $u(x, t)$.

It remains to prove the initial condition. It has to be understood in a limiting sense because the formula itself has meaning only for $t > 0$. Because the integral of S is 1, we have

$$\begin{aligned}
u(x,t) - \phi(x) &= \int_{-\infty}^{\infty} S(x-y,t)\,[\phi(y) - \phi(x)]\,dy \\
&= \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{\infty} e^{-p^2/4} [\phi(x - p\sqrt{kt}) - \phi(x)]\,dp.
\end{aligned}$$

For fixed x we must show that this tends to zero as $t \to 0$. The idea is that for $p\sqrt{t}$ small, the continuity of ϕ makes the integral small; while for $p\sqrt{t}$ not small, p is large and the exponential factor is small.

To carry out this idea, let $\epsilon > 0$. Let $\delta > 0$ be so small that

$$\max_{|y-x| \le \delta} |\phi(y) - \phi(x)| < \frac{\epsilon}{2}.$$

This can be done because ϕ is continuous at x. We break up the integral into the part where $|p| < \delta/\sqrt{kt}$ and the part where $|p| \geq \delta/\sqrt{kt}$. The first part is

$$\left|\int_{|p|<\delta/\sqrt{kt}}\right| \leq \left(\frac{1}{\sqrt{4\pi}}\int e^{-p^2/4}dp\right) \cdot \max_{|y-x|\leq\delta}|\phi(y)-\phi(x)|$$
$$< 1\cdot\frac{\epsilon}{2} = \frac{\epsilon}{2}.$$

The second part is

$$\left|\int_{|p|\geq\delta/\sqrt{kt}}\right| \leq \frac{1}{\sqrt{4\pi}}\cdot 2(\max|\phi|)\cdot\int_{|p|\geq\delta/\sqrt{kt}} e^{-p^2/4}dp < \frac{\epsilon}{2}$$

by choosing t sufficiently small, since the integral $\int_{-\infty}^{\infty} e^{-p^2/4}\,dp$ converges and δ is fixed. (That is, the "tails" $\int_{|p|\geq N} e^{-p^2/4}\,dp$ are as small as we wish if $N = \delta/\sqrt{kt}$ is large enough.) Therefore,

$$|u(x,t)-\phi(x)| < \tfrac{1}{2}\epsilon + \tfrac{1}{2}\epsilon = \epsilon$$

provided that t is small enough. This means exactly that $u(x,t) \to \phi(x)$ as $t \to 0$. $\square$

Corollary. The solution has all derivatives of all orders for $t > 0$, even if ϕ is not differentiable. We can say therefore that all solutions become smooth as soon as diffusion takes effect. There are no singularities, in sharp contrast to the wave equation.

Proof. We use formula (1)

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,dy$$

together with the rule for differentiation under an integral sign, Theorem 2 in Section A.3.

Piecewise Continuous Initial Data. Notice that the continuity of $\phi(x)$ was used in only one part of the proof. With an appropriate change we can allow $\phi(x)$ to have a jump discontinuity. [Consider, for instance, the initial data for $Q(x,t)$.]

A function $\phi(x)$ is said to have a *jump* at x_0 if both the limit of $\phi(x)$ as $x \to x_0$ from the right exists [denoted $\phi(x_0+)$] and the limit from the left [denoted $\phi(x_0-)$] exists but these two limits are not equal. A function is called *piecewise continuous* if in each finite interval it has only a finite number of jumps and it is continuous at all other points. This concept is discussed in more detail in Section 5.2.

Theorem 2. Let $\phi(x)$ be a bounded function that is piecewise continuous. Then (1) is an infinitely differentiable solution for $t > 0$ and

$$\boxed{\lim_{t \searrow 0} u(x,t) = \tfrac{1}{2}[\phi(x+) + \phi(x-)]}$$

for all x. At every point of continuity this limit equals $\phi(x)$.

Proof. The idea is the same as before. The only difference is to split the integrals into $p > 0$ and $p < 0$. We need to show that

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \phi(x + \sqrt{kt}\, p)\, dp \to \pm \frac{1}{2} \phi\,(x\pm).$$

The details are left as an exercise. □

EXERCISES

1. Prove that if ϕ is any piecewise continuous function, then

$$\frac{1}{\sqrt{4\pi}} \int_0^{\pm\infty} e^{-p^2/4} \phi(x + \sqrt{kt}\, p)\, dp \to \pm \frac{1}{2} \phi\,(x\pm) \quad \text{as } t \searrow 0.$$

2. Use Exercise 1 to prove Theorem 2.