Real Analysis II

Chapter 9 Sequences and Series of Functions

9.1 Pointwise Convergence of Sequence of Functions

Definition 9.1 A Let $\{f_n\}$ be a sequence of functions defined on a set of real numbers E. We say that $\{f_n\}$ **converges pointwise to a function** f **on** E for each $x \in E$, the sequence of real numbers $\{f_n(x)\}$ converges to the number $f(x)$. In other words, for each $x \in E$, we have

$$\lim_{x\to\infty} f_n(x) = f(x).$$

Example 1) Let

$$f_n(x) = x^n, \qquad x \in [0,1]$$

and let

$$f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1. \end{cases}$$

Then $\{f_n\}$ converges to f pointwise on $[0,1]$.

2) Let

$$g_n(x) = \frac{x}{1+nx}, \qquad x \in [0,\infty]$$

Then $\{g_n\}$ converges to $g(x) = 0$ pointwise on $[0,\infty]$.

3) Let

$$h_n(x) = \frac{nx}{1+n^2x^2}, \qquad x \in [0,\infty]$$

Then $\{h_n\}$ converges to $h(x) = 0$ pointwise on $[0,\infty]$.

4) Let

$$\chi_n(x) = \begin{cases} 1 & \text{if } x \in [-n,n] \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{\chi_n\}$ converges to $\chi(x) = 1$ pointwise on $[-\infty,\infty]$.

Remark. Suppose $\{f_n\}$ converges pointwise to f on E. Then given $\epsilon > 0$, and given $x \in E$, there exists $N = N(x,\epsilon) \in \mathbf{I}$, such that

$$|f_n(x) - f(x)| < \epsilon, \qquad \text{for all } n \ge N.$$

In general N depends on ϵ as well as x.

For example, consider the sequence of Example 1 above: $f_(x) = x^n$ and $f(x) = 0$, $(0 \le x < 1)$ and $f(1) = 1$. If $\epsilon = 1/2$, then, for each $x \in [0,1]$, there exists N such that

$$|f_n(x) - f(x)| \le \frac{1}{2}. \qquad \text{for all } \quad (n \ge N) \qquad (*)$$

For $x = 0$ or $x = 1$, then $(*)$ holds with $N = 1$. For $x = 3/4 = 0.75$, $(*)$ holds with $N = 3$ and for $x = 0.9$ we need $N = 7$.

We claim that there is no N for which $(*)$ hold for all $x \in [0,1]$. For if there is such an N, then for all $x \in [0,1)$, (*) implies

$$x^n < \frac{1}{2}.$$

In particular we would have

$$x^N < \frac{1}{2}$$

for all $x \in [0,1)$. Taking limit as $x \to 1^-$ we would have $1 \le 1/2$, which is a contradiction.

If g_n is as given in Example 2 above :

$$g_n(x) = \frac{x}{1+nx},$$

then we have

$$g_n(x) \le \frac{1}{n}$$

for all $x \in [0,\infty)$ and hence for a given $\epsilon > 0$, any N with $N > 1/\epsilon$ will imply that

$$|g_n(x) - 0| < \epsilon \qquad \text{for all } \; n > N \quad \text{and for all} \quad x \in [0,\infty).$$

We leave to you to analyze the situations for the sequences in Examples 3 and 4 above.

9.2 Uniform Convergence of Sequence of Functions

Definition 9.2A Let $\{f_n\}$ be a sequence of functions on E. We say that $\{f_n\}$ **converges uniformly to** f **on** E if for given $\epsilon > 0$, there exists $N = N(\epsilon)$, depending on ϵ only, such that

$$|f_n(x) - f(x)| < \epsilon \qquad \text{for all } \; n > N \quad \text{and for all} \quad x \in E.$$

If $\{f_n\}$ converges to f uniformly to f on E, we write

$$f_n \to f \quad \text{uniformly on} \;\; E.$$

Remark. 1) Unlike the pointwise converges, in the case of uniform convergence, we note N depends only on ϵ and not on x.
2) If $f_n \to f$ uniformly on E, then $f_n \to f$ pointwise on E. The sequence $f_n(x) = x^n$ on $[0,1]$ discussed in Example 1 of the previous section shows that the converse of the above statement is not true.

Example The sequence

$$g_n(x) = \frac{x}{1+nx}$$

converges uniformly to 0 on $[0,\infty)$. It is a good exercise to show whether the sequences of Examples 3 and 4 of the previous section are uniformly convergent or not.

The following corollary is a restatement of the definition of uniform converges. It is useful to show that a sequence is not uniformly convergent.

Corollary 9.2B The sequence $\{f_n\}$ does *not* converge uniformly to f on E if and only if there exists an $\epsilon > 0$ such that there is no $N > 0$ for which

$$|f_n(x) - f(x)|, \epsilon \quad \text{for all } \; n > N \;\text{ for all } x \in E$$

holds.

Remark 1) If $f_n \to f$ uniformly on E and $\epsilon > 0$, then there exists $N > 0$ such that for all $n > N$, the entire graph of $y = f_n(x)$ lies between the graphs of $y = f(x) - \epsilon$ and $y = f(x) + \epsilon$.

2) If $f_n \to 0$ uniformly on E and $\epsilon > 0$, then there exists $N > 0$ such that for all $n > N$ and all $x \in E$, $|f_n(x)| < \epsilon$. This implies that for all $n > N$,

$$\sup_{x\in E} |f_n(x)| \le \epsilon \qquad \text{and hence} \qquad \lim_{n\to\infty} \sup_{x\in E} |f_n(x)| \le \epsilon.$$

Since $\epsilon > 0$ is an arbitrary positive number, we conclude that

$$\text{If } f_n \to 0 \text{ uniformly on } E, \text{ then } \lim_{n\to\infty} \sup_{x\in E} |f_n(x)| = 0.$$

The converse is also true and the proof is an exercise.

Example For the sequence

$$h_n(x) = \frac{nx}{1 + n^2x^2}$$

we have

$$\sup_{x\in[0,\infty)} |h_n(x)| \ge h_n\left(\frac{1}{n}\right) = \frac{1}{2} \qquad \text{and hence} \qquad \lim_{n\to\infty} \sup_{x\in[0,\infty)} |h_n(x)| \ne 0.$$

Thus $\{h_n\}$ does not converge uniformly to 0 on $[0, \infty)$.

Theorem 9.2E

$$f_n \to f \quad \text{uniformly on } E \text{ if and only if } \lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0.$$

Theorem 9.2F (Cauchy Criterion for Uniform Convergence) A sequence $\{f_n\}$ converges uniformly on E if and only if for a given $\epsilon > 0$, there exists $N > 0$ such that for all $n \ge m > N$ and for all $x \in E$,

$$|f_n(x) - f_m(x)| < \epsilon.$$

Theorem 9.2G If $\{f_n\}$ is a sequence of continuous functions on a bounded and closed interval $[a, b]$ and $\{f_n\}$ converges pointwise to a continuous function f on $[a, b]$, then $f_n \to f$ uniformly on $[a, b]$.

9.3 Consequences of Uniform Convergence

Theorem 9.3A If $f_n \to f$ uniformly on $[a, b]$, if f_n are continuous at $c \in [a, b]$, then f is continuous at c.

Corollary 9.3B If $f_n \to f$ uniformly on $[a, b]$, if f_n are continuous on $[a, b]$, then f is continuous on $[a, b]$.

Remark Does $f_n \in \mathcal{R}[a, b]$ and $f \to f$ pointwise on $[a, b]$ imply that $f \in \mathcal{R}[a, b]$? The answer is no. For example, let

$$A = \{r_1,\ r_2,\ r_3,\ \cdots\}$$

be the set of all rational numbers in $[0, 1]$, and let

$$A_n = \{r_1,\ r_2,\ \cdots,\ r_n\}.$$

Let χ_n be the characteristic function of A_n and χ be the characteristic of A. Since χ_n is discontinuous only at a finite number of points (where ?), we see that $\chi_n \in \mathcal{R}[a, b]$. On the other hand, χ is not continuouos at any point in $[0, 1]$ and hence $\chi \notin \mathcal{R}[a, b]$. Clearly $\chi_n \to \chi$ on $[0, 1]$ pointwise.

Theorem 9.3E If $f_n \in \mathcal{R}[a,b]$ and if $f_n \to f$ uniformly on $[a,b]$, then $f \in \mathcal{R}[a,b]$.

Remark In Theorems 9.3A and 9.3E, uniform convergence is sufficient. The sequence $h_n(x) = \frac{nx}{1+n^2x^2}$ for $x \in [0,\infty)$, converges to the continuous function $h(x) = 0$. Recall that the converges is not uniform. The sequence $f_n(x) = x^n$ on $[0,1]$ can be used to show that uniform convergence is not necessary for theorem 9.3E (explain).

Remark When does $f_n \to f$ imply $\int_a^b f \to \int_a^b f$? To answer this question, we consider the following example. Let

$$f_n(x) = \begin{cases} 2n & \text{if } \frac{1}{n} \le x \le \frac{2}{n} \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\int_0^1 f_n(x)\,dx = \int_{\frac{1}{n}}^{\frac{2}{n}} 2n\,dx = 2n\left(\frac{2}{n} - \frac{1}{n}\right) = 2 \qquad \text{and hence} \qquad \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 2.$$

On the other hand, for fixed $x \in [0,1]$, we can choose an N so that $x > 2/N$ and hence $f_n(x) = 0$ for all $n \ge N$. Therefore

$$f_n \to 0 \quad \text{pointwise and hence} \quad \int_0^1 \lim_{n\to\infty} f_n(x)\,dx = 0.$$

Theorem 9.3G If $f_n \in \mathcal{R}[a,b]$ and if $f_n \to f$ uniformly on $[a,b]$, then

$$\int_a^b f \to \int_a^b f.$$

Remark Let $f_n(x) = \frac{x^n}{n}$ on $[0,1]$ and let $f(x) = 0$. Then $f_n \to f$ uniformly but $f_n'(1) = 1$ while $f'(1) = 0$. Thus

$$\lim_{n\to\infty} f_n'(x) = f'(x)$$

does not hold at $x = 1$.

Theorem 9.3I If $f_n'(x)$ exists for each n and each $x \in [a,b]$, if f_n' is continuous on $[a,b]$, if $\{f_n\}$ convegres uniformly to f on $[a,b]$, and if $\{f_n'\}$ convegres uniformly to g on $[a,b]$, then $g = f'$.

9.4 Convergence and Uniform Convergence of Series of Functions

Definiton 9.4A Let $\{u_n\}$ be a sequence of functions and let $s_n(x) = \sum_{k=1}^n u_k(x)$ be the nth partial sum of the infintie series $\sum_{k=1}^\infty u_k(x)$. We say $\sum_{k=1}^\infty u_k$ converges pointwise to f on E if $s_n \to f$ pointwise on E. In this case we write

$$\sum_{k=1}^\infty u_k = f \quad \text{pointwise on } E$$

Example Let $u_k = x^k$, $-1 < x < 1$ and let $f(x)\frac{x}{1-x}$. Then

$$\sum_{k=1}^\infty u_k \;=\; f \quad \text{pointwise on } (-1,1).$$

Definiton 9.4B We say that $\sum_{k=1}^\infty u_k$ converges to f uniformly on E if $s_n \to f$ uniformly on E. We write

$$\sum_{k=1}^\infty u_k \;=\; f \quad \text{uniformly on } E$$

Theorem 9.4C If $\sum_{k=1}^{\infty} u_k \ = f$ uniformly on E, and if $\{u_k\}$ is continuous on E, then f is continuous on E.

Exnaple Let

$$u_n(x) = x(1-x^n), \quad (0 \leq x \leq 1, \quad n = 0, 1, 2, \cdots), \quad \text{and let} \quad f(x) = \begin{cases} 1, & \text{if } 0 < x \leq 1 \\ 0, & \text{if } x = 0. \end{cases}$$

Then $\sum u_n \ = \ f$ pointwise on $[0, 1]$.(Verify this) Clearly f is not continuous at $x = 0$ while u_n is continuous for each n.

Theorem 9.4E (Weierstrass M-Test) If $\{u_k\}$ is a sequence of continuous functions such that $|u_k(x)| \leq M_k$ for all $x \in E$ and if $\sum M_k$ is convergent, then

$$\sum_{k=1}^{\infty} u_k \ = f \quad \text{uniformly on } E$$

Notation if $\{a_k\}$ and $\{b_k\}$ are two sequences, and if $a_k \leq b_k$, we write

$$\sum_{k=1}^{\infty} a_k << \sum_{k=1}^{\infty} b_k$$

Thus Weierstrass' Theorem states that

$$\text{if } \sum_{k=1}^{\infty} u_k << \sum_{k=1}^{\infty} M_k < \infty, \quad \text{then, for some function} f, \quad \sum_{k=1}^{\infty} u_k \ = f \quad \text{uniformly on } E$$

Example Since

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2} << \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

Weierstrass'Theorem implies that

$$\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$$

converges uniformly on $\mathbf{R}$.

Theorem 9.4F If the power series $\sum_{k=0}^{\infty} a_k x^k$ converges for $x = x_0$ (with $x_0 \neq 0$), then the power series converges uniformly on $[-x_1, x_1]$ for any $x_1 \in [0, |x_0|]$.

Theorem 9.4G Let $\{u_k\}$ is a sequence of continouous nonnegative functions on $[a, b]$ and if $\sum_{k=0}^{\infty} u_k$ convgres pointwise to a continuous function f on $[a, b]$, then

$$\sum_{k=1}^{\infty} u_k \ = f \quad \text{uniformly on } [a, b].$$

9.5 Integration and Differentiation of Series of Functions

Theorem 9.5A Let $\{u_k\}$ be sequence of fucntions in $\mathcal{R}[a,b]$. and suppose $\sum_{k=1}^{\infty} u_k = f$ uniformly on $[a,b]$. Then $f \in \mathcal{R}[a,b]$ and

$$\int_a^b f(x)\,dx = \int_a^b \left(\sum_{k=1}^{\infty} u_k(x)\right)\,dx = \sum_{k=1}^{\infty}\left(\int_a^b u_k(x)\,dx\right)$$

Theorem 9.5B If $\{u_k\}$ is differentiable on $[a,b]$, if $\{u_k'\}$ is continuous on $[a,b]$, if $\sum u_k = f$ uniformly, and if $\sum u_k'$ converges uniformly on $[a,b]$, then

$$\sum_{k=1}^{\infty} u_k'(x) \;=\; f'(x)$$

Example 1)

$$1 - x + x^2 - x^3 \;+\; \cdots \;=\; \frac{1}{1+x} \qquad \text{uniformly on } \;(0,1)$$

implies that for any $y \in (0,1)$,

$$\int_0^y 1dx - \int_0^y xdx + \int_0^y x^2dx - \int_0^y x^3dx \;+\; \cdots \;=\; \int_0^y \frac{1}{1+x}dx$$

from which we conclude that

$$y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots = \log(1+y).$$

2)

$$1 + x + x^2 + x^3 \;+\; \cdots \;=\; \frac{1}{1-x} \qquad \text{uniformly on } \;(0,1)$$

implies that for all $x \in (-1,1)$,

$$1 + 2x + 3x^2 + 4x^3 \;+\; \cdots \;=\; \frac{1}{(1-x)^2}.$$

Theorem 9.5C If the power series $\sum_{k=0}^{\infty} a_k x^k$ convegres to $f(x)$ on $[-b,b]$ for some $b > 0$, then for any $x \in [-b,b]$,

$$f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

Corollary 9.5D

$$\text{If } \; f(x) = \sum_{k=0}^{\infty} a_k x^k, \quad \text{then} \qquad f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n+1)a_k x^{k-n}.$$

Example (A Continuous Nowhere Differentiable Function)

Let $f_0(x) =$ the distance from x to the nearest integer. (Thus $f_0(0.45) = 0.45$ and $f_0(3.67) = 0.33$) Define $f_k(x) = f_k(10^k x)$ and

$$F(x) = \sum_{k=0}^{\infty} \frac{f_k(x)}{10^k}.$$

Then F is continouos everywhere and differentiable nowhere.

Example Another example of everywhere continuous and nowhere differentiable function is due to Weierstrass and is given by

$$G(x) = \sum_{k=0}^{\infty} \frac{\cos(3^n x)}{2^n}.$$