SEQUENCES AND SERIES OF FUNCTIONS: I

PETE L. CLARK

1. Pointwise Convergence and Cautionary Tales

Consider a sequence of functions $\{f_n : [a,b] \rightarrow \mathbb{R}\}_{n=1}^{\infty}$. What would it mean for f_n to converge to a function f?

Note that for each point x in $[a,b]$, plugging in x we get a sequence of numbers $\{f_n(x)\}$. It seems reasonable to define $f_n \rightarrow f$ to mean that for each x that we plug in, the sequence of numbers $f_n(x)$ converges to $f(x)$. More formally:

Definition: Let $\{f_n\}$ be a sequence of functions and f be any function. [1] We say that f_n **converges pointwise** to f if for every x in the domain, $f_n(x) \rightarrow f(x)$.

We also say that f is the **pointwise limit** of the sequence $\{f_n\}$. Note that the pointwise limit, if it exists, is uniquely determined: it is just the function $x \mapsto \lim_{n\rightarrow\infty} f_n(x)$.

In a similar way we can define a series of functions converging (or summing) pointwise to a function f, namely $\sum_{n=1}^{\infty} f_n(x) = f$ means: $S_n(x) \rightarrow f$ pointwise, where $S_n(x) := f_1(x) + \ldots + f_n(x)$.

The highest form of praise that we have for a numerical sequence is "convergent", and similarly the most we could expect out of a numerical series is for it to be absolutely convergent. The situation is much more interesting for sequences and series of functions: if each of the f_n's has some nice property (like integrability, continuity or differentiability) and $f_n \rightarrow f$, it is natural to wonder whether the limit function f retains this nice property.

Especially, we are interested in the following "example": suppose $\{a_n\}_{n=0}^{\infty}$ is some sequence of numbers, and define $f_n(x) = a_n x^n$. Then $\sum_{n=0}^{\infty} f_n = \sum_{n=0}^{\infty} a_n x^n$ is a power series. Notice that the Nth partial sum, $\sum_{n=0}^{N} a_n x^n$, is just a polynomial, and as such it has all kinds of nice properties: for instance it is infinitely differentiable. We saw that when the quantity $\overline{\theta} \limsup |a_n^{\frac{1}{n}}| < \infty$, then taking $R = \frac{1}{\overline{\theta}}$, the power series converges at least for x in the open interval $(-R,R)$. We would very much like such functions – i.e., functions defined by convergent power series – to be nice functions: as we saw earlier in the course, if only it turns out to be true that $f = \sum_n a_n x^n$ is itself infinitely differentiable, and if it is the case that $f' = \sum_n f_n' = \sum_n n a_n x^{n-1}$, then we can work with the (vastly larger) class of

[1] It is implicit that the f_n's and f have a common domain, say a closed interval $[a,b]$ to fix ideas. The common domain could also be something like $(-\infty,b)$, (a,∞) or $(-\infty,\infty)$.

functions given by power series in much the same way as we work with polynomials.

Thus it is very natural to ask the following questions: suppose that $f_n \to f$ is a sequence of functions converging pointwise to a limit function f. Then:
a) If each f_n is Riemann-integrable, must f be Riemann-integrable?
a') If so, is $\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$?
b) If each f_n is continuous, must f be continuous?
c) If each f_n is differentiable, must f be differentiable?
c') If all the f_n's and f are differentiable, is $f' = \lim_{n\to\infty} f_n'$?

Before attempting to give answers, let us note that all of these questions are really questions of the form: can the order of two limiting processes be interchanged? For instance, since $f = \lim_{n\to\infty} f_n$, part a') is really asking: if we take the $\lim_{n\to\infty}$ first and then do the $\int$, will we get the same answer as if we took the $\int$ followed by the $\lim_{n\to\infty}$? Similarly, asking whether f is continuous is asking whether, for each $c \in [a, b]$ it is true that $\lim_{x\to c} f(x) = f(c)$, and since $f = \lim_{n\to\infty}$ we are asking about the validity of the following equation:

$$f(c) = \lim_{n\to\infty} \lim_{x\to c} f_n(x) \stackrel{?}{=} \lim_{x\to c} \lim_{n\to\infty} f_n(x) = \lim_{x\to c} f(x).$$

The first and the last equalities certainly hold (by definition of f), so what is at issue is the apparently innocuous swapping of the order of the two limits. Similar remarks apply to the differentiation question, since this is also a limiting operation.

Remember two lessons we learned from our study of infinite series: once we give a formal definition of a limiting processes we have to be careful in not assuming that the consequences of this definition are necessarily what we would intuitively expect. (For instance, we found that the product of two convergent series is not always a convergent series; that changing the order of the terms can affect the sum and even the convergence; and that even regrouping the sum by adding parentheses can make a divergent series converge. In summary, none of the distributive, the commutative, or associative laws for ordinary finite sums behaved as we would have initially expected.) However, we did not just give up and go home: we found that most of the facts that we wanted to be true about series are true, provided we work with some additional hypotheses (in the case of series, the magic hypothesis was absolute convergence).

It turns out that the same (rather strange) phenomenon crops up again for sequences (and series) of functions: the answer to each of these questions is "NO!" However we will not give up; rather we will find additional conditions that suffice to ensure that the answer is "YES." In particular, these hypotheses will hold for power series on the interior of the interval of convergence.

First let us give a very simple example to deflate our hopes that "lim's" can be interchanged willy-nilly.

Example 1: Consider $a_{m,n} = \frac{m}{m+n}$. We let m and n both range over all positive integers. It is easy to see that

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{mn} \neq \lim_{n\to\infty}\lim_{m\to\infty} a_{mn}.$$

Indeed, for any fixed m, $\lim_{n\to\infty} \frac{m}{m+n} = 0$, and since $\lim_{m\to\infty} 0 = 0$, we get that $\lim_{m\to\infty}\lim_{n\to\infty} a_{mn} = 0$. On the other hand, for any fixed n, $\lim_{m\to\infty} \frac{m}{m+n} = \lim_{m\to\infty} \frac{1}{1+\frac{n}{m}} \to \frac{1}{1+0} = 1$, so $\lim_{n\to\infty}\lim_{m\to\infty} a_{mn} = \lim_{n\to\infty} 1 = 1$.

Now you may say to yourself, "Wow, scary. But I don't care about limits of double sequences." Okay, but consider the following:

Example 2: Let $f_n = x^n : [0,1] \to \mathbb{R}$. Of course each f_n is continuous (indeed, infinitely differentiable). However, for any $x \in [0,1)$, $\lim_{n\to\infty} x^n = 0$, while $\lim_{n\to\infty} 1^n = 1$. Thus the limit function is the function which is identically zero on $[0,1)$ and jumps up to 1 at $x = 1$, so is discontinuous. Thus the limit of even such a simple sequence of continuous functions can be discontinuous.

Example 3: Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of the rational numbers in the unit interval $[0,1]$. We define a sequence of functions $f_n : [0,1] \to \mathbb{R}$ as follows: if x is irrational then $f_n(x) = 0$ for all n. Otherwise, $x = r_k$ for a unique k and we define $f_n(r_k)$ to be equal to 0 if $k > n$ and to be equal to 1 if $k \leq n$. In other words, the function f_n is nonzero precisely at $r_1, r_2, \ldots, r_n$, and at these numbers it takes the value 1. Since each f_n is discontinuous precisely at these n points $r_1, \ldots, r_n$ and is clearly bounded, it is Riemann-integrable. However, we claim that f_n converges pointwise to the function which is 0 at every irrational number and 1 at every rational value; this is our prototypical example of a function which is not Riemann integrable. For every irrational x, $f_n(x) = 0$ for all x, and the zero sequence converges to zero. [2] If $x = r_k$, then for all $n \geq k$, $f_n(r_k) = 1$, so the sequence is ultimately constant, hence converging to 1.

Example 4: Let $f_n : [0,1] \to \mathbb{R}$ be defined as follows: $f_n(x) = n$ for $x \in (0, 1/n)$ and $f_n(x) = 0$ otherwise. (Draw a picture!) Since f_n is bounded and discontinuous only at the finite set of points $\{0, \frac{1}{n}\}$, it is integrable. Moreover, $\int_0^1 f_n = 1$, the area of a rectangle with width $\frac{1}{n}$ and height n. We claim that $f_n \to 0$ pointwise: indeed $f_n(0) = 0$ for all n and for any $x > 0$, for all n such that $\frac{1}{n} < x$ (i.e., for all sufficiently large n) we have $f_n(x) = 0$. However, clearly $\int_0^1 0 = 0$, so despite the fact that the limit function is integrable, we have

$$0 = \int_0^1 \lim_{n\to\infty} f_n \neq \lim_{n\to\infty} \int_0^1 f_n = 1.$$

We could continue by giving many more examples of bad behavior of pointwise limits. For instance, the limit function in Example 2 was discontinuous, but not *too* discontinuous in the cosmic scheme of things: it has only a simple discontinuity at a single point, so in particular it is a *regulated* function. But this need not be the case, and in fact if we think carefully we already know there are worse examples than this:

[2] "Nothing can come of nothing..." – King Lear

Example 5: We can construe the differentiation process as passage to a pointwise limit. Namely, suppose that $f : [a, b] \to \mathbb{R}$ is a differentiable function. Put $f_n(x) = n(f(x + \frac{1}{n}) - f(x)) = \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}$. Since as $n \to \infty$, $\frac{1}{n} \to 0$, $\lim_{n\to\infty} f_n(x) = f'(x)$. Note that since f is differentiable it is continuous and, being built out of continuous functions in a suitable way, each f_n is clearly continuous (in fact differentiable). But recall that a derivative f' need not be continuous (e.g. $f(x) = x \sin(\frac{1}{x})$). In the cited example the right-hand limit of f' as f approaches zero fails to exist. This gives an example of a pointwise limit of continuous (even differentiable) functions not being regulated.

Exercise 73*:
a) Suppose $f : [a, b] \to \mathbb{R}$ is a regulated function which satisfies the Intermediate Value Property: for any subinterval $[c, d]$ of $[a, b]$ and any number L in between $f(c)$ and $f(d)$, there exists a point $x \in (c, d)$ such that $f(x) = L$. Show that f is continuous.
b) Conclude from Darboux's Theorem that a derivative which is not continuous fails to be regulated.

2. Introducing Uniform Convergence

Let us not give up too easily. It turns out that a slightly stronger form of convergence is enough to give positive answers to many of these questions.

Definition: Let $\{f_n\}$ be a sequence of functions with domain $[a, b]$. We say f_n **converges uniformly** to f and write $f_n \xrightarrow{u} f$ if: for all $\epsilon > 0$, there exists an N depending only on ϵ such that for all $n \geq N$ and all $x \in [a, b]$, $|f_n(x) - f(x)| < \epsilon$.

How does this definition differ from the definition of pointwise convergence? The answer is subtle but critical: we say that $f_n \to f$ pointwise if for all $x \in [a, b]$, $f_n(x) \to f(x)$. This in turn means that for any fixed x and all $\epsilon > 0$, there exists an N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$. The only difference is that since in the definition of pointwise convergence we are concerned only with one value of x at a time, the N we choose is thus allowed to depend not only on ϵ but also on the point x itself. In the definition of uniform convergence, there must exist a single N which makes $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$. Thus uniform convergence is a astronger condition than pointwise convergence (and implies it).

One could be forgiven for thinking that this small stengthening – namely, of requring N to be independent of x – would not make much difference. In light of Examples 2 and 5, the following results show that it makes all the difference in the world. After we prove them, we will come back for a closer look at how the two definitions really differ.

Theorem 1. *Let f_n be a sequence of functions defined on $[a, b]$, and let c be a point of $[a, b]$. Assume that for every n, $\lim_{x\to c} f_n = L_n$. Suppose also that $f_n \xrightarrow{u} f$. Then the sequence L_n is convergent, and its limit L is equal to $\lim_{x\to c} f$.*

Remarks: In other words, the Theorem is asserting precisely that, under the hypothesis of *uniform* convergence, we have

$$\lim_{x\to c}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\lim_{x\to c} f_n(x).$$

In other words, assuming uniform convergence the two limits can be interchanged.

The statement of the theorem is a bit complicated; in applications one uses mostly the following consequence.

Corollary 2. *Suppose that f_n is a sequence of continuous functions and $f_n \xrightarrow{u} f$. Then f is continuous.*

Exercise 74: Prove Corollary 2. (This is immediate from the various definitions; the point is to make you go over them and understand them.)

Proof of Theorem 1: Note that, as usual, there is a Cauchy criterion for uniform convergence: $f_n \xrightarrow{u} f$ if and only if: for all $\epsilon > 0$ there exists $N = N(\epsilon)$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all x in $[a, b]$.

The first real step is to show that the sequence L_n is convergent. Since we don't know what we're trying to show it converges to, we should try to prove that it's Cauchy instead, so we consider the difference $|L_n - L_m|$. Now comes the trick: for any $x \in [a, b]$ we have

$$|L_n - L_m| \le |L_n - f_n(x)| + |f_n(x) - f_m(x)| + |f_m(x) - L_m|.$$

The middle term can be made small precisely by the Cauchy condition for uniform convergence: specifically, we choose N such that $n, m \ge N$ implies $|f_n(x)-f_m(x)| < \frac{\epsilon}{3}$ for all x in the domain. Now we look at the first and last terms and see that, by definition of L_n and L_m, they too can be made small by taking x sufficiently close to c: there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$ then $|L_n - f_n(x)| < \frac{\epsilon}{3}$ and $|L_m - f_m(x)| < \frac{\epsilon}{3}$. This gives that $|L_n - L_m| < \epsilon$ for all $m, n \ge N$, so that the sequence $\{L_n\}$ is Cauchy and hence convergent, say to L.

The second step is to show that $\lim_{x\to c} f(x) = L$ and this is remarkably similar:

$$|f(x) - L| \le |f(x) - f_n(x)| + |f_n(x) - L_n| + |L_n - L|.$$

Since $L_n \to L$ and $f_n(x) \to f(x)$, the first and last term will each be less than $\frac{\epsilon}{3}$ when n is sufficiently large. Since $f_n(x) \to L_n$, the middle term will be less than $\frac{\epsilon}{3}$ for x sufficiently close to c. Overall this shows $|f(x)-L| < \epsilon$ for x sufficiently close to c, i.e., that $\lim_{x\to c} f(x) = L$, completing hte proof of the theorem.

While we are here, let's prove one more result about uniform convergence.

Theorem 3. *Let f_n be a sequence of integrable functions on $[a, b]$. Assume that $f_n \xrightarrow{u} f$. Then f is integrable and moreover,*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof: Fix $\epsilon > 0$ and choose N such that $n \ge N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$. Now let P be any partition of $[a, b]$. Since $|f(x)| \le |f_n(x)| + \epsilon$, we get that

$$U(f, P) \le U(f_n, P) + U(\epsilon, P) = U(f_n, P) + (b - a)\epsilon$$

and similarly

$$L(f,P) \geq L(f_n,P) + L(-\epsilon,P) = L(f_n,P) - (b-a)\epsilon.$$

But this means that $\omega(f,P) \leq \omega(f_n,P) + 2(b-a)\epsilon$. By the Darboux integrability criterion, we can choose P so that the first term is arbitrarily small. Thus $\overline{\int}_a^b f - \underline{\int}_a^b f$ is bounded by a quantity that goes to zero as $n \to \infty$, so f is (Darboux) integrable. Since we showed in particular that the difference between the upper sums $U(f,P)$ and $U(f_n,P)$ goes to zero as n approaches infinity, we must have that $\int_a^b f_n \to \int_a^b f$. This completes the proof of the theorem.

3. Uniform Convergence and Differentiability

In light of Corollary 2 and Theorem 3, one might expect that if $f_n \xrightarrow{u} f$, and each f_n is differentiable, then f is differentiable and $f' = \lim_{n\to\infty} f_n'$. However, this is not the case: differentiability is a much more delicate property that continuity or integrability. We give two different counterexamples.

Example 6: Let $f(x) : \mathbb{R} \to \mathbb{R}$ be a function which is bounded, differentiable, and such that $\lim_{n\to\infty} f'(n)$ does not exist; for instance, one could take $f(x) = \sin(\frac{\pi x}{2})$. Define

$$f_n(x) := \frac{1}{n} f(nx).$$

Then $f_n \xrightarrow{u} 0$: indeed, if $|f(x)| \leq M$ for all n, then $|f_n(x)| < \frac{M}{n}$. Thus, taking n to be sufficiently large such that $\frac{M}{n} < \epsilon$, we get $|f_n(x) - 0| < \epsilon$ for all real x. Evidently the limit function, $f(x) \equiv 0$, is differentiable. However, $f_n'(x) = \frac{1}{n}\left(n \cdot f_{\prime(nx))=f'(nx)}\right.$, so $\lim_{n\to\infty} f_n'(1) = \lim_{n\to\infty} f'(n)$, which does not exist by hypothesis. Thus, although the uniform limit is differentiable, the sequence of derivatives $f_n'(x)$ does not converge at $x = 1$.

Example 7: More fundamentally, I claim that one cannot tell whether a continuous function is differentiable by looking at its graph: the differentiability of a continuous function can be gained (or lost) by arbitarily small variations.

To see what we are getting at, we will find a sequence of differentiable functions $f_n : [-1,1] \to \mathbb{R}$ which converge uniformly to $f(x) = |x|$. Essentially, we will "round off the corner" at $x = 0$ by making smaller and smaller changes. Indeed, the function f_n will be equal to $f(x) = |x|$ except on the open interval $(\frac{-1}{n}, \frac{1}{n})$ in which range it will be given by some polynomial $P_n(x)$. We want to choose $P_n(x)$ so that the piecewise-defined function $f(x)$ is differentiable at $x = \pm\frac{1}{n}$; in order to do this we need to require that $P_n(\frac{1}{n}) = |\frac{1}{n}| = \frac{1}{n}$, $P_n(-\frac{1}{n}) = |\frac{-1}{n}| = \frac{1}{n}$, $P_n'(\frac{1}{n}) = 1$ and $P_n'(\frac{-1}{n}) = \frac{-1}{n}$. (In other words, we need the values of P_n and its derivative at $x = \pm\frac{1}{n}$ to be equal to the corresponding values of $|x|$ and its derivative at these points.) We will also require $P_n(0) = 0$, and search for the simplest polynomial function that satisfies these properties.

In fact we have five independent conditions on $P_n(x)$, so there will be a unique degree four polynomial satisfying all these conditions. To see what it is, we should in principle write out an arbitrary degree four polynomial and solve five linear equations for the five unknown coefficients of P. However, we can cut down our work a bit by noticing that all our conditions are compatible with $P_n(x)$ being an *even* function of x, which for a quartic polynomial means that it is of the form

$$P_n(x) = Ax^4 + Bx^2 + C.$$

Since we want $P_n(0) = 0$, this gives $C = 0$. We now have only two unknowns to solve for, using the two conditions

$$\frac{1}{n} = P_n(\frac{1}{n}) = A(\frac{1}{n})^4 + B(\frac{1}{n})^2$$

and

$$1 = P_n'(\frac{1}{n}) = A(\frac{1}{n})^3 + 2B\frac{1}{n}.$$

This system is easy to solve (for instance, multiply the first equation by 4, multiply the second equation by $\frac{1}{n}$, and then subtract the second equation from the first to eliminate B): we get $A = \frac{-1}{2}n^3$ and $B = \frac{3}{2}n$, so

$$P_n(x) = \frac{-1}{2}n^3x^4 + \frac{3}{2}nx^2.$$

Recall that we have defined a function $f_n(x)$ by:
$f_n(x) = P_n(x)$, if $x \in [-\frac{1}{n}, \frac{1}{n}]$.
$f_n(x) = |x|$, if $x \in [-1,1] \setminus [-\frac{1}{n}, \frac{1}{n}]$.
We claim that $f_n \xrightarrow{u} |x|$; since all the f_n's and $f(x) = |x|$ are even, by symmetry we may show uniform convergence on $[0,1]$: thus we need to show that or in other words that $f_n - x \xrightarrow{u} 0$. Fix $\epsilon > 0$; we will find an N such that $n \geq N$ implies $|f_n(x) - x| < \epsilon$ for all $x \in [0,1]$. Note that since $f_n(x) = x$ unless $x < \frac{1}{n}$, in estimating the difference between $f_n(x)$ and x we may assume that $x \leq \frac{1}{n}$, or $nx \leq 1$. In this case we have

$$|f_n(x) - x| = |P_n(x) - x| = |x||\frac{-1}{2}(nx)^3 + \frac{3}{2}(nx)|.$$

Using the estimate $nx \leq 1$, this quantity is at most $2|x| \leq \frac{2}{n}$. In other words, we've shown that $|f_n(x) - x| \leq \frac{1}{n}$ for all x on $[0,1]$, and this shows that $f_n - x \xrightarrow{u} 0$.

Remarks: a) As we will see at the very end of the course, a much stronger result is true: for any continuous function $f : [a,b] \to \mathbb{R}$, there exists a sequence of **polynomials** P_n such that $P_n \xrightarrow{u} f$ on $[a,b]$. This is Weierstrass's **Approximation Theorem**, and it is justly remembered as one of the greatest results of one of the great analysts of all time. This is an incredibly deep result, even given – as we are – the benefit of hindsight. As far as I know there is no really conceptually straightforward proof of this result, and in analyzing this example we are closer than you might expect to the heart of the matter: it turns out that finding a sequence of polynomials $P_n(x)$ converging uniformly to $|x|$ on $[-1,1]$ is half of the proof of Weierstrass' theorem! Note well that we did not acheive this goal here: our functions $f_n(x)$ are not polynomials (in fact $f_n(x)$ fails to be twice differentiable at $x = \pm\frac{1}{n}$, as you are invited to check).

b) As we may (or may not) see later on, there are continuous functions which are not differentiable at any point: this result is also due to Weierstrass. In fact even more is true: in a sense that we will unfortunately not be able to make precise in this course, "most" continuous functions $f : [a,b] \to \mathbb{R}$ are not differentiable at any point.

Nevertheless there is still a result about convergence of derivatives: we just need a stronger hypothesis.

Theorem 4. *Let f_n be a sequence of differentiable functions on $[a,b]$ such that $f_n(x_0)$ converges for some point x_0 of $[a,b]$. If f_n' converges uniformly on $[a,b]$, then f_n converges uniformly on $[a,b]$ to a function f, and moreover we have $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a,b]$.*

Proof:[3] Fix $\epsilon > 0$,and choose N such that $m, n \geq N$ implies $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$ and $|f_n'(t) - f'm(t)| < \frac{\epsilon}{2(b-a)}$ for all $t \in [a,b]$. This latter estimate can be viewed as telling us that the derivative of $g := f_n - f_m$ is small on the entire interval $[a,b]$; applying the Mean Value Theorem to g, we get

$$|g(x) - g(t)| = |x-t||f'(c)| \leq |x-t| \cdot \frac{\epsilon}{2(b-a)} \leq \frac{\epsilon}{2}, \tag{1}$$

valid for all $x,\ t \in [a,b]$ and all $m,\ n \geq N$. Now for all $x \in [a,b]$

$$|g(x)| \leq |g(x) - g(x_0)| + |g(x_0)| < \epsilon.$$

This shows that the sequence f_n is uniformly Cauchy, hence uniformly converges to some function, say f, on $[a,b]$. Now **fix** an x in $[a,b]$ and define $\varphi_n(t) := \frac{f_n(t)-f_n(x)}{t-x}$, $\varphi(t) := \frac{f(t)-f(x)}{t-x}$, the point of this being that $\lim_{x\to t}\varphi_n(t) = f_n'(x)$ (for any n). Now the first inequality in (1) shows that

$$|\varphi_n(t) - \varphi_m(t)| \leq \frac{\epsilon}{2(b-a)}$$

for all $m, , n \geq N$, so that φ_n converges uniformly for all $t \neq x$. Since $f_n \to f$, by definition of φ_n and φ we get $\varphi_n \xrightarrow{u} \varphi$ for all $t \neq x$. Now we can apply Theorem 1 on the interchange of limit operations:

$$f'(x) = \lim_{t\to x}\varphi(t) = \lim_{t\to x}\lim_{n\to\infty}\varphi_n(t) = \lim_{n\to\infty}\lim_{t\to x}\varphi_n(t) = \lim_{n\to\infty} f_n'(x),$$

completing the proof of the theorem.

Remark: This was, of course, not an easy proof. You should compare the result with Theorem 7.12 in your text. The version we give is stronger for two reasons: first, we require f_n to converge to f only at a single point. The second, and more interesting, difference is that Theorem 7.12 in your text assumes that the f_n''s are *continuous*. If we assume this additional hypothesis, then the proof becomes much easier: it is enough to know that if $f_n \xrightarrow{u} f$ is a uniformly convergent sequence of functions on $[a,b]$, then defining antiderivatives $F_n(x) := \int_a^x f_n$ and $F(x) := \int_a^x f$, then $F_n \xrightarrow{u} F$. Then: if $f_n' \xrightarrow{u} f'$ and f_n' is continuous for all n, then f' is necessarily continuous and $\int_a^x f_n' \xrightarrow{u} \int_a^x f'$ and you can use the Fundamental Theorem of Calculus to get the desired conclusion. In fact the argument will again only use the pointwise convergence of f_n to f at a single point (namely a). You are asked to fill in the details in the exercises.

Thus the real point of going the long way around is not to assume that the derivatives f_n' are continuous. (In fact the easier argument would work even if f_n' were merely assumed to be Riemann-integrable, but as we saw, not all derivatives

[3]Taken from Rudin's *Principles of Mathematical Analysis*, pp. 152-3. Note this is stronger than the version which appears in your text.

have this property.) It is a general theme in analysis that functions with *continuous* derivatives are very well-behaved, whereas things are much more delicate for arbitrary differentiable functions.

4. Some criteria for uniform convergence

We are now convinced that the right notion of convergence of a sequence of functions is uniform convergence. It would therefore be nice to have some general criteria by which we can show that a particular sequence of functions is uniformly convergent. We present several of these; the most useful by far is Weierstrass' criterion for uniform convergence of series of functions.

First a definition: let I be an interval (perhaps infinite) and let $f : I \to \mathbb{R}$ be any function. Let us define the norm of f, $||f||$, to be $\sup_{x \in I} ||f||$. Note that $||f|| \in [0, +\infty]$: it could be infinite. In fact, to say that $||f|| < \infty$ is precisely to say that f is bounded, in which case $||f||$ is just the least M which works as a bound for f. So there is really nothing new here except the notation.

However, the notation is useful: for instance, we find that the statement "$f_n \xrightarrow{u} f$" is equivalent to: $||f_n - f|| \to 0$. (We should be a little careful about one point: $||f_n - f||$ could, for any given n be infinite; what we mean by the convergence to zero is that it is finite for all sufficiently large n and from this point on the sequence of ordinary real numbers converges to zero.) Although I swore to the department chair that I would not use the words "m∃tric sp@ces" in this course, you are encouraged to think of $||f_n - f||$ as the distance between the functions f_n and f.

Theorem 5. *(Weierstrass M-Test) Let f_k be a sequence of functions defined on an interval I. Suppose that $\sum_{k=1}^{\infty} ||f_k|| < \infty$. Then $\sum_{k=1}^{\infty} f_k$ is uniformly convergent.*

Proof: Put $S_n(x) = \sum_{k=1}^{n} f_k(x)$ and $M_k := ||f_k||$. Then, what we are really assuming is that $\sum_{k=1}^{\infty} M_k = M < \infty$ (hence the rather silly name "M Test"), so in particular the sequence of partial sums is Cauchy: for all $\epsilon > 0$, there exists $N = N(\epsilon)$ such that for all $n \geq N$ and all $m \geq 1$ we have

$$S_{n+m} - S_n = \sum_{k=n+1}^{n+m} M_k < \epsilon.$$

But this means that for all $x \in I$,

$$|S_{n+m}(x) - S_n(x)| = |f_{n+1}(x) + \ldots + f_{n+m}(x)| \leq |f_{n+1}(x)| + \ldots + |f_{n+m}(x)|$$
$$\leq M_{n+1} + +M_{n+m} < \epsilon,$$

so $S_n(x)$ is uniformly convergent by the Cauchy criterion.

As a simple application of this result, suppose that $f_n(x)$ is any sequence of functions with $||f_n|| \leq 1$ for all n (i.e., each is bounded by 1) and $\sum_{n=1}^{\infty} a_n$ is any absolutely convergent real series. Then the series of functions $\sum_{n=1}^{\infty} a_n f_n$ is uniformly convergent by the M Test. For instance, take $\{g_n(x)\}$ to be *any* sequence of function and take $f_n(x) = \sin(g_n(x))$. Then $\sum_{n=1}^{\infty} \frac{\sin(g_n(x))}{n^2}$ is uniformly convergent.

Your text contains a further result which is interesting but (honestly) much less useful. To state it, we first need a little terminology: by an **increasing sequence** of functions $\{f_n\}$ defined on an interval I, we mean a sequence such that, for all $x \in I$, $f_n(x) \leq f_{n+1}(x)$ for all n. Caution: this is not the same as a sequence of increasing functions: rather, the point is that for fixed x, as we increase n the function values get larger and larger. For instance, $f_n(x) = \sin x - \frac{1}{n}$ is an increasing sequence of functions, although each f_n is certainly not an increasing function.

Theorem 6. *(Dini's Theorem) Let $\{f_n\}$ be an increasing sequence of continuous functions defined on a closed interval $[a, b]$. If the sequence is pointwise convergent, it converges uniformly.*

Exercise 75*: Prove Dini's Theorem. (Hint: consult Exercise 6 in §7.3 of your text, which breaks up the proof into a sequence of steps. Even with all this coaching, this is still not an easy exercise.)