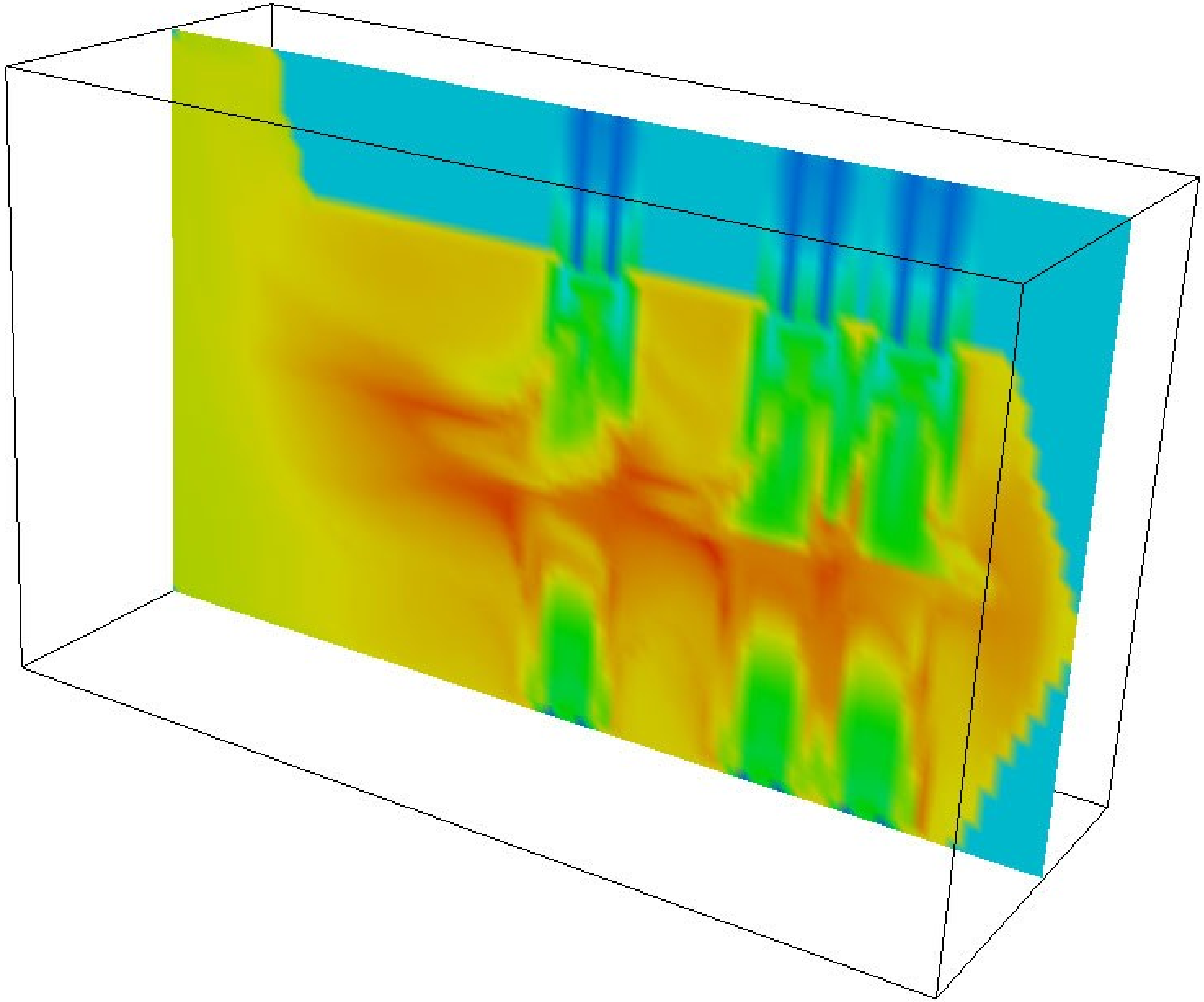

Resolução das equações de Navier-Stokes utilizando Transformada Rápida de Fourier

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Nosso objeto de estudo

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \mu\nabla^2\vec{u} + \vec{f}(\vec{x},t)$$

$$\nabla\cdot\vec{u} = 0$$

Discretizamos!

- Domínio periódico
- Tamanho L
- Discretize em N pontos equi-espaçados (malha estruturada)
- $h=L/N$

Operadores de diferenças

$$(D_\alpha^0 \phi)(\vec{x}) = \frac{\phi(\vec{x} + h\vec{e_\alpha}) - \phi(\vec{x} - h\vec{e_\alpha})}{2h}$$

$$\vec{D}^0 = (D_1^0, D_2^0, D_3^0)$$

Operadores de diferenças

$$(D_\alpha^+\phi)(\vec{x}) = \frac{\phi(\vec{x} + h\vec{e_\alpha}) - \phi(\vec{x})}{h}$$

$$(D_\alpha^-\phi)(\vec{x}) = \frac{\phi(\vec{x}) - \phi(\vec{x} - h\vec{e_\alpha})}{h}$$

Relacionando…

$$\vec{D}^0 \to \nabla$$

$$\sum_{\alpha=1}^{3} D_\alpha^+ D_\alpha^- \to \nabla^2$$

Um pouco diferente

$$u_\alpha D_\alpha^{\pm} = \begin{cases} u_\alpha D_\alpha^{+}, u_\alpha < 0 \\ u_\alpha D_\alpha^{-}, u_\alpha > 0 \end{cases}$$

Um pouco diferente

$$u_\alpha D_\alpha^\pm = \begin{cases} u_\alpha D_\alpha^+, u_\alpha < 0 \\ u_\alpha D_\alpha^-, u_\alpha > 0 \end{cases}$$

$$\Delta t \sum_{\alpha=1}^{3} | \, u_\alpha^n(\vec{x}) \, | \leq h$$

As equações contínuas...

$$\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \mu\nabla^2\vec{u} + \vec{f}(\vec{x},t)$$

$$\nabla\cdot\vec{u} = 0$$

... e discretizadas!

$$\rho\left(\frac{\vec{u}^{n+1}-\vec{u}^{n}}{\Delta t}+\sum_{\alpha=1}^{3} u_{\alpha}^{n} D_{\alpha}^{\pm}\vec{u}^{n}\right)=-\vec{D}^{0}p^{n+1}+\mu\sum_{\alpha=1}^{3} D_{\alpha}^{+}D_{\alpha}^{-}\vec{u}^{n+1}+\vec{f}$$

$$\vec{D}^{0}\cdot\vec{u}^{n+1}=0$$

Reorganizando

$$\left(1 - \frac{\mu \Delta t}{\rho} \sum_{\alpha=1}^{3} D_\alpha^+ D_\alpha^-\right) \vec{u}^{n+1} + \frac{\Delta t}{\rho} \vec{D}^0 p^{n+1} = \vec{v}^n$$

$$\vec{v}^n = \vec{u}^n - \frac{\Delta t}{\rho} \sum_{\alpha=1}^{3} u_\alpha^n D_\alpha^\pm \vec{u}^n + \frac{\Delta t}{\rho} \vec{f}^n$$

$$\vec{D}^0 \cdot \vec{u}^{n+1} = 0$$

A Transformada

$$\hat{\phi}(\vec{k}) = \frac{1}{L^3}\sum_{\vec{x}} \phi(\vec{x}) h^3 exp\left(-i2\pi\frac{\vec{k}\cdot\vec{x}}{L}\right)$$

Transformada de um operador

$$\hat{D}_\alpha^+ \hat{D}_\alpha^- = -\frac{4}{h^2}\left(sin\frac{k_\alpha \pi}{N}\right)^2$$

$$\hat{\vec{D}}^0 = \frac{i}{h} sin(\frac{2\pi}{N}\vec{k})$$

No espaço de Fourier...

$$A(\vec{k}) = 1 + \frac{4\mu\Delta t}{\rho h^2} \sum_{\alpha=1}^{3} \left(sin\frac{k_\alpha \pi}{N} \right)^2$$

$$A(\vec{k})\hat{\vec{u}}^{n+1}(\vec{k}) + \frac{i\Delta t}{\rho h} sin\left(\frac{2\pi}{N}\vec{k}\right)\hat{p}^{n+1}(\vec{k}) = \hat{\vec{v}}^{n}(\vec{k})$$

$$\frac{i}{h} sin\left(\frac{2\pi}{N}\vec{k}\right) \cdot \hat{\vec{u}}^{n+1} = 0$$

Resolvendo!

$$\hat{p}^{n+1} = -\frac{i\Delta t}{\rho h} \frac{(sin\frac{2\pi}{N}\vec{k}) \cdot \hat{\vec{v}}^{n}(\vec{k})}{(sin\frac{2\pi}{N}\vec{k}) \cdot (sin\frac{2\pi}{N}\vec{k})}$$

$$\hat{\vec{u}}^{n+1}(\vec{k}) = \frac{\hat{\vec{v}}^{n}(\vec{k}) - \frac{i\Delta t}{\rho h}(sin\frac{2\pi}{N}\vec{k})\hat{p}^{n+1}}{A(\vec{k})}$$

Resolvendo!

$$\hat{p}^{n+1} = -\frac{i\Delta t}{\rho h}\frac{(sin\frac{2\pi}{N}\vec{k}) \cdot \hat{\vec{v}}^n(\vec{k})}{(sin\frac{2\pi}{N}\vec{k}) \cdot (sin\frac{2\pi}{N}\vec{k})}$$

Pode ser zero! Neste caso, p=0

$$\hat{\vec{u}}^{n+1}(\vec{k}) = \frac{\hat{\vec{v}}^n(\vec{k}) - \frac{i\Delta t}{\rho h}(sin\frac{2\pi}{N}\vec{k})\hat{p}^{n+1}}{A(\vec{k})}$$

Algoritmo

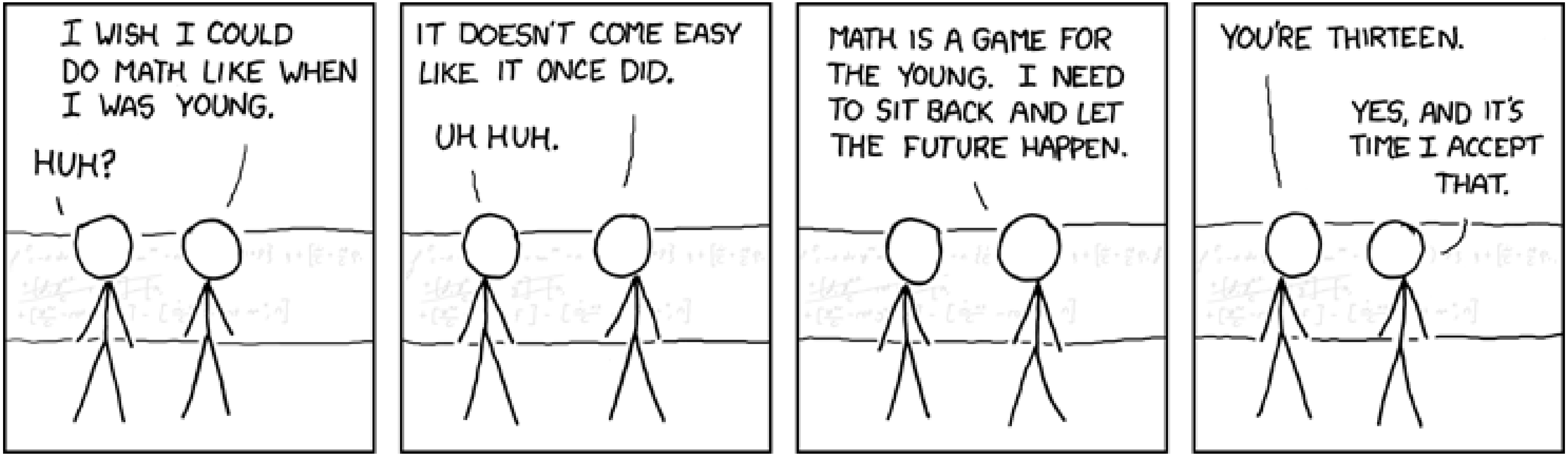
- ENTRADA: Termo forçante (f), parâmetros físicos e computacionais, condição inicial (U e p no tempo 0)
- SAÍDA: Condição final (U e p no tempo T)

Algoritmo

- Para cada passo temporal:
  - Determine Dt
  - Calcule v
  - Transforme U, p, v
  - Resolva p(n+1) e U(n+1) no espaço de Fourier
  - Transforme U e p (inversa)

Referência

- Peskin, C.S. & McQueen, D.M. **Fluid Dynamics of the Heart and its Valves**. In: Othmer et al, Case Studies in Mathematical Modeling – Ecology, Physiology, and Cell Biology. Prentice-Hall, 1996.



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Obrigado!