# Numerical Analysis

SEVENTH EDITION

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| Time | 0 | 3 | 5 | 8 | 10 | 13 |
|---|---|---|---|---|---|---|
| Distance | 0 | 225 | 383 | 623 | 742 | 993 |

**20.** In a circuit with impressed voltage $\mathcal{E}(t)$ and inductance $L$, Kirchhoff's first law gives the relationship

$$\mathcal{E}(t) = L\frac{di}{dt} + Ri,$$

where $R$ is the resistance in the circuit and $i$ is the current. Suppose we measure the current for several values of $t$ and obtain:

| $t$ | 1.00 | 1.01 | 1.02 | 1.03 | 1.0 |
|---|---|---|---|---|---|
| $i$ | 3.10 | 3.12 | 3.14 | 3.18 | 3.24 |

where $t$ is measured in seconds, $i$ is in amperes, the inductance $L$ is a constant 0.98 henries, and the resistance is 0.142 ohms. Approximate the voltage $\mathcal{E}(t)$ when $t = 1.00$, 1.01, 1.02, 1.03, and 1.04.

**21.** All calculus students know that the derivative of a function $f$ at $x$ can be defined as

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Choose your favorite function $f$, nonzero number $x$, and computer or calculator. Generate approximations $f'_n(x)$ to $f'(x)$ by

$$f'_n(x) = \frac{f(x+10^{-n}) - f(x)}{10^{-n}},$$

for $n = 1, 2, \ldots, 20$, and describe what happens.

**22.** Derive a method for approximating $f'''(x_0)$ whose error term is of order $h^2$ by expanding the function $f$ in a fourth Taylor polynomial about $x_0$ and evaluating at $x_0 \pm h$ and $x_0 \pm 2h$.

**23.** Consider the function

$$e(h) = \frac{\varepsilon}{h} + \frac{h^2}{6}M,$$

where $M$ is a bound for the third derivative of a function. Show that $e(h)$ has a minimum at $\sqrt[3]{3\varepsilon/M}$.

## 4.2 Richardson's Extrapolation

Richardson's extrapolation is used to generate high-accuracy results while using low-order formulas. Although the name attached to the method refers to a paper written by L. F. Richardson and J. A. Gaunt [RG] in 1927, the idea behind the technique is much older. An interesting article regarding the history and application of extrapolation can be found in [Joy].

Extrapolation can be applied whenever it is known that an approximation technique has an error term with a predictable form, one that depends on a parameter, usually the step size $h$. Suppose that for each number $h \neq 0$ we have a formula $N(h)$ that approximates an unknown value $M$ and that the truncation error involved with the approximation has the form



$$M - N(h) = K_1h + K_2h^2 + K_3h^3 + \cdots,$$

for some collection of unknown constants $K_1, K_2, K_3, \ldots$.

Since the truncation error is $O(h)$, we would expect, for example, that

$$M - N(0.1) \approx 0.1K_1, \quad M - N(0.01) \approx 0.01K_1,$$

and, in general, $M - N(h) \approx K_1h$, unless there was a large variation in magnitude among the constants $K_1, K_2, K_3, \ldots$.

The object of extrapolation is to find an easy way to combine the rather inaccurate $O(h)$ approximations in an appropriate way to produce formulas with a higher-order truncation error. Suppose, for example, we could combine the $N(h)$ formulas so as to produce an $O(h^2)$ approximation formula, $\hat{N}(h)$, for $M$ with

$$M - \hat{N}(h) = \hat{K}_2h^2 + \hat{K}_3h^3 + \cdots,$$

for some, again unknown, collection of constants $\hat{K}_1, \hat{K}_2, \ldots$. Then we would have

$$M - \hat{N}(0.1) \approx 0.01\hat{K}_2, \quad M - \hat{N}(0.01) \approx 0.0001\hat{K}_2,$$

and so on. If the constants $K_1$ and $\hat{K}_2$ are roughly of the same magnitude, then the $\hat{N}(h)$ approximations would be much better than the corresponding $N(h)$ approximations. The extrapolation continues by combining the $\hat{N}(h)$ approximations in a manner that produces formulas with $O(h^3)$ truncation error, and so on.

To see specifically how we can generate these higher-order formulas, let us consider the formula for approximating $M$ of the form

$$M = N(h) + K_1h + K_2h^2 + K_3h^3 + \cdots. \tag{4.10}$$

Since the formula is assumed to hold for all positive $h$, consider the result when we replace the parameter $h$ by half its value. Then we have the formula

$$M = N\left(\frac{h}{2}\right) + K_1\frac{h}{2} + K_2\frac{h^2}{4} + K_3\frac{h^3}{8} + \cdots.$$

Subtracting (4.10) from twice this equation eliminates the term involving $K_1$ and gives

$$M = \left[N\left(\frac{h}{2}\right) + \left(N\left(\frac{h}{2}\right) - N(h)\right)\right] + K_2\left(\frac{h^2}{2} - h^2\right) + K_3\left(\frac{h^3}{4} - h^3\right) + \cdots.$$

To facilitate the discussion, we define $N_1(h) \equiv N(h)$ and

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \left[N_1\left(\frac{h}{2}\right) - N_1(h)\right].$$

Then we have the $O(h^2)$ approximation formula for $M$:

$$M = N_2(h) - \frac{K_2}{2}h^2 - \frac{3K_3}{4}h^3 - \cdots. \tag{4.11}$$

If we now replace $h$ by $h/2$ in this formula, we have

$$M = N_2\left(\frac{h}{2}\right) - \frac{K_2}{8}h^2 - \frac{3K_3}{32}h^3 - \cdots. \tag{4.12}$$

This can be combined with Eq. (4.11) to eliminate the $h^2$ term. Specifically, subtracting (4.11) from 4 times Eq. (4.12) gives

$$3M = 4N_2\left(\frac{h}{2}\right) - N_2(h) + \frac{3K_3}{8}h^3 + \cdots,$$

and dividing by 3 gives an $O(h^3)$ formula for approximating $M$:

$$M = \left[N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{3}\right] + \frac{K_3}{8}h^3 + \cdots.$$

By defining

$$N_3(h) \equiv N_2\left(\frac{h}{2}\right) + \frac{N_2(h/2) - N_2(h)}{3},$$

we have the $O(h^3)$ formula:

$$M = N_3(h) + \frac{K_3}{8}h^3 + \cdots.$$

The process is continued by constructing an $O(h^4)$ approximation

$$N_4(h) = N_3\left(\frac{h}{2}\right) + \frac{N_3(h/2) - N_3(h)}{7},$$

an $O(h^5)$ approximation

$$N_5(h) = N_4\left(\frac{h}{2}\right) + \frac{N_4(h/2) - N_4(h)}{15},$$

and so on. In general, if $M$ can be written in the form

$$M = N(h) + \sum_{j=1}^{m-1} K_j h^j + O(h^m), \tag{4.13}$$

then for each $j = 2, 3, \ldots, m$, we have an $O(h^j)$ approximation of the form

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{2^{j-1} - 1}. \tag{4.14}$$

These approximations are generated by rows in the order indicated by the numbered entries in Table 4.5. This is done to take best advantage of the highest-order formulas.

Extrapolation can be applied whenever the truncation error for a formula has the form

$$\sum_{j=1}^{m-1} K_j h^{\alpha_j} + O(h^{\alpha_m}),$$



**Table 4.5**

| $O(h)$ | $O(h^2)$ | $O(h^3)$ | $O(h^4)$ |
|---|---|---|---|
| **1:** $N_1(h) \equiv N(h)$ | | | |
| **2:** $N_1(\frac{h}{2}) \equiv N(\frac{h}{2})$ | **3:** $N_2(h)$ | | |
| **4:** $N_1(\frac{h}{4}) \equiv N(\frac{h}{4})$ | **5:** $N_2(\frac{h}{2})$ | **6:** $N_3(h)$ | |
| **7:** $N_1(\frac{h}{8}) \equiv N(\frac{h}{8})$ | **8:** $N_2(\frac{h}{4})$ | **9:** $N_3(\frac{h}{2})$ | **10:** $N_4(h)$ |

for a collection of constants $K_j$ and when $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_m$. In the next example we have $\alpha_j = 2j$.

**EXAMPLE 1** The centered difference formula in Eq. (4.5) to approximate $f'(x_0)$ can be expressed with an error formula:

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \cdots.$$

Since this error formula contains only even powers of $h$, extrapolation is more effective than as outlined in the opening discussion. In this case, we have the $O(h^2)$ approximation

$$f'(x_0) = N_1(h) - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(x_0) - \cdots, \tag{4.15}$$

where

$$N_1(h) \equiv N(h) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)].$$

Replacing $h$ by $h/2$ in this formula gives the approximation

$$f'(x_0) = N_1\left(\frac{h}{2}\right) - \frac{h^2}{24}f'''(x_0) - \frac{h^4}{1920}f^{(5)}(x_0) - \cdots.$$

Subtracting (4.15) from 4 times this equation eliminates the $O(h^2)$ term that involves $f'''(x_0)$ and gives

$$3f'(x_0) = 4N_1\left(\frac{h}{2}\right) - N_1(h) + \frac{h^4}{160}f^{(5)}(x_0) + \cdots.$$

Dividing by 3 provides an $O(h^4)$ formula

$$f'(x_0) = N_2(h) + \frac{h^4}{480}f^{(5)}(x_0) + \cdots,$$

where

$$N_2(h) = N_1\left(\frac{h}{2}\right) + \frac{N_1(h/2) - N_1(h)}{3}.$$

Continuing this procedure gives, for each $j = 2, 3, \ldots$, an $O(h^{2j})$ approximation

$$N_j(h) = N_{j-1}\left(\frac{h}{2}\right) + \frac{N_{j-1}(h/2) - N_{j-1}(h)}{4^{j-1} - 1}.$$

Notice that the denominator of the quotient is $4^{j-1}-1$ instead of $2^{j-1}-1$ because we are now eliminating powers of $h^2$ instead of powers of $h$. Since $(h/2)^2 = h^2/4$, the multipliers used to eliminate the powers of $h^2$ are powers of 4 instead of 2.

Suppose that $x_0 = 2.0$, $h = 0.2$, and $f(x) = xe^x$. Then

$$N_1(0.2) = N(0.2) = \frac{1}{0.4}[f(2.2) - f(1.8)] = 22.414160,$$
$$N_1(0.1) = N(0.1) = 22.228786,$$

and

$$N_1(0.05) = N(0.05) = 22.182564.$$

The extrapolation table for these data is shown in Table 4.6. The exact value of $f'(x) = xe^x + e^x$ at $x_0 = 2.0$ to six decimal places is 22.167168, so all the digits of $N_3(0.2)$ are exact, even though the best original approximation, $N_1(0.05)$, has only one decimal place of accuracy. ■

**Table 4.6**

| | | |
|---|---|---|
| $N_1(0.2) = 22.414160$ | | |
| $N_1(0.1) = 22.228786$ | $N_2(0.2) = N_1(0.1) + \frac{N_1(0.1) - N_1(0.2)}{3} = 22.166995$ | |
| $N_1(0.05) = 22.182564$ | $N_2(0.1) = N_1(0.05) + \frac{N_1(0.05) - N_1(0.1)}{3} = 22.167157$ | $N_3(0.2) = N_2(0.1) + \frac{N_2(0.1) - N_2(0.2)}{15} = 22.167168$ |

Since each column beyond the first in the extrapolation table is obtained by a simple averaging process, the technique can produce high-order approximations with minimal computational cost. However, as $k$ increases, the roundoff error in $N_1(h/2^k)$ will generally increase because the instability of numerical differentiation is related to the step size $h/2^k$.

In Section 4.1, we discussed both three- and five-point methods for approximating $f'(x_0)$ given various functional values of $f$. The three-point methods were derived by differentiating a Lagrange interpolating polynomial for $f$. The five-point methods can be obtained in a similar manner, but the derivation is tedious. Extrapolation can be used to derive these formulas more easily.

Suppose we expand the function $f$ in a fourth Taylor polynomial about $x_0$. Then

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{6}f'''(x_0)(x-x_0)^3 + \frac{1}{24}f^{(4)}(x_0)(x-x_0)^4 + \frac{1}{120}f^{(5)}(\xi)(x-x_0)^5,$$

for some number $\xi$ between $x$ and $x_0$. Evaluating $f$ at $x_0 + h$ and $x_0 - h$ gives



$$f(x_0+h) = f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 + \frac{1}{120}f^{(5)}(\xi_1)h^5 \quad (4.16)$$

and

$$f(x_0-h) = f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 - \frac{1}{6}f'''(x_0)h^3 + \frac{1}{24}f^{(4)}(x_0)h^4 - \frac{1}{120}f^{(5)}(\xi_2)h^5, \quad (4.17)$$

where $x_0 - h < \xi_2 < x_0 < \xi_1 < x_0 + h$. Subtracting Eq. (4.17) from Eq. (4.16) produces

$$f(x_0+h) - f(x_0-h) = 2hf'(x_0) + \frac{h^3}{3}f'''(x_0) + \frac{h^5}{120}[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)]. \quad (4.18)$$

If $f^{(5)}$ is continuous on $[x_0 - h, x_0 + h]$, the Intermediate Value Theorem implies that a number $\tilde{\xi}$ in $(x_0 - h, x_0 + h)$ exists with

$$f^{(5)}(\tilde{\xi}) = \frac{1}{2}[f^{(5)}(\xi_1) + f^{(5)}(\xi_2)].$$

As a consequence, Eq. (4.18) can be solved for $f'(x_0)$ to give the $O(h^2)$ approximation

$$f'(x_0) = \frac{1}{2h}[f(x_0+h) - f(x_0-h)] - \frac{h^2}{6}f'''(x_0) - \frac{h^4}{120}f^{(5)}(\tilde{\xi}). \quad (4.19)$$

Although the approximation in Eq. (4.19) is the same as that given in the three-point formula in Eq. (4.5), the unknown evaluation point occurs now in $f^{(5)}$, rather than in $f'''$. Extrapolation takes advantage of this by first replacing $h$ in Eq. (4.19) with $2h$ to give the new formula

$$f'(x_0) = \frac{1}{4h}[f(x_0+2h) - f(x_0-2h)] - \frac{4h^2}{6}f'''(x_0) - \frac{16h^4}{120}f^{(5)}(\hat{\xi}), \quad (4.20)$$

where $\hat{\xi}$ is between $x_0 - 2h$ and $x_0 + 2h$.

Multiplying Eq. (4.19) by 4 and subtracting Eq. (4.20) produces

$$3f'(x_0) = \frac{2}{h}[f(x_0+h) - f(x_0-h)] - \frac{1}{4h}[f(x_0+2h) - f(x_0-2h)] - \frac{h^4}{30}f^{(5)}(\tilde{\xi}) + \frac{2h^4}{15}f^{(5)}(\hat{\xi}).$$

If $f^{(5)}$ is continuous on $[x_0 - 2h, x_0 + 2h]$, an alternative method can be used to show that $f^{(5)}(\tilde{\xi})$ and $f^{(5)}(\hat{\xi})$ can be replaced by a common value $f^{(5)}(\xi)$. Using this result and dividing by 3 produces the five-point formula

$$f'(x_0) = \frac{1}{12h}[f(x_0-2h) - 8f(x_0-h) + 8f(x_0+h) - f(x_0+2h)] + \frac{h^4}{30}f^{(5)}(\xi),$$

which is the five-point formula given as Eq. (4.6). Other formulas for first and higher derivatives can be derived in a similar manner. Some of these are considered in the exercises.

The technique of extrapolation is used throughout the text. The most prominent applications occur in approximating integrals in Section 4.5 and for determining approximate solutions to differential equations in Section 5.8.

## EXERCISE SET 4.2

1. Apply the extrapolation process described in Example 1 to determine $N_3(h)$, an approximation to $f'(x_0)$, for the following functions and stepsizes.
   a. $f(x) = \ln x$, $x_0 = 1.0$, $h = 0.4$
   b. $f(x) = x + e^x$, $x_0 = 0.0$, $h = 0.4$
   c. $f(x) = 2^x \sin x$, $x_0 = 1.05$, $h = 0.4$
   d. $f(x) = x^3 \cos x$, $x_0 = 2.3$, $h = 0.4$
2. Add another line to the extrapolation table in Exercise 1 to obtain the approximation $N_4(h)$.
3. Repeat Exercise 1 using four-digit rounding arithmetic.
4. Repeat Exercise 2 using four-digit rounding arithmetic.
5. The following data give approximations to the integral

$$M = \int_0^{\pi} \sin x \, dx.$$

$$N_1(h) = 1.570796, \quad N_1\left(\frac{h}{2}\right) = 1.896119, \quad N_1\left(\frac{h}{4}\right) = 1.974232, \quad N_1\left(\frac{h}{8}\right) = 1.993570.$$

   Assuming $M = N_1(h) + K_1h^2 + K_2h^4 + K_3h^6 + K_4h^8 + O(h^{10})$, construct an extrapolation table to determine $N_4(h)$.
6. The following data can be used to approximate the integral

$$M = \int_0^{3\pi/2} \cos x \, dx.$$

$$N_1(h) = 2.356194, \quad N_1\left(\frac{h}{2}\right) = -0.4879837,$$

$$N_1\left(\frac{h}{4}\right) = -0.8815732, \quad N_1\left(\frac{h}{8}\right) = -0.9709157.$$

   Assume a formula exists of the type given in Exercise 5 and determine $N_4(h)$.
7. Show that the five-point formula in Eq. (4.6) applied to $f(x) = xe^x$ at $x_0 = 2.0$ gives $N_2(0.2)$ in Table 4.6 when $h = 0.1$ and $N_2(0.1)$ when $h = 0.05$.
8. The forward-difference formula can be expressed as

$$f'(x_0) = \frac{1}{h}[f(x_0+h) - f(x_0)] - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

   Use extrapolation to derive an $O(h^3)$ formula for $f'(x_0)$.
9. Suppose that $N(h)$ is an approximation to $M$ for every $h > 0$ and that

$$M = N(h) + K_1h + K_2h^2 + K_3h^3 + \cdots,$$



   for some constants $K_1, K_2, K_3, \ldots$. Use the values $N(h)$, $N\left(\frac{h}{3}\right)$, and $N\left(\frac{h}{9}\right)$ to produce an $O(h^3)$ approximation to $M$.
10. Suppose that $N(h)$ is an approximation to $M$ for every $h > 0$ and that

$$M = N(h) + K_1h^2 + K_2h^4 + K_3h^6 + \cdots,$$

   for some constants $K_1, K_2, K_3, \ldots$. Use the values $N(h)$, $N\left(\frac{h}{3}\right)$, and $N\left(\frac{h}{9}\right)$ to produce an $O(h^6)$ approximation to $M$.
11. In calculus, we learn that $e = \lim_{h\to 0}(1+h)^{1/h}$.
    a. Determine approximations to $e$ corresponding to $h = 0.04$, $0.02$, and $0.01$.
    b. Use extrapolation on the approximations, assuming that constants $K_1, K_2, \ldots$, exist with $e = (1+h)^{1/h} + K_1h + K_2h^2 + K_3h^3 + \cdots$, to produce an $O(h^3)$ approximation to $e$, where $h = 0.04$.
    c. Do you think that the assumption in part (b) is correct?
12. a. Show that

$$\lim_{h\to 0}\left(\frac{2+h}{2-h}\right)^{1/h} = e.$$

    b. Compute approximations to $e$ using the formula $N(h) = \left(\frac{2+h}{2-h}\right)^{1/h}$, for $h = 0.04$, $0.02$, and $0.01$.
    c. Assume that $e = N(h) + K_1h + K_2h^2 + K_3h^3 + \cdots$. Use extrapolation, with at least 16 digits of precision, to compute an $O(h^3)$ approximation to $e$ with $h = 0.04$. Do you think the assumption is correct?
    d. Show that $N(-h) = N(h)$.
    e. Use part (d) to show that $K_1 = K_3 = K_5 = \cdots = 0$ in the formula

$$e = N(h) + K_1h + K_2h^2 + K_3h^3K_4h^4 + K_5h^5 + \cdots,$$

    so that the formula reduces to

$$e = N(h) + K_2h^2 + K_4h^4 + K_6h^6 + \cdots.$$

    f. Use the results of part (e) and extrapolation to compute an $O(h^6)$ approximation to $e$ with $h = 0.04$.
13. Suppose the following extrapolation table has been constructed to approximate the number $M$ with $M = N_1(h) + K_1h^2 + K_2h^4 + K_3h^6$:

| | | |
|---|---|---|
| $N_1(h)$ | | |
| $N_1\left(\frac{h}{2}\right)$ | $N_2(h)$ | |
| $N_1\left(\frac{h}{4}\right)$ | $N_2\left(\frac{h}{2}\right)$ | $N_3(h)$ |

    a. Show that the linear interpolating polynomial $P_{0,1}(h)$ through $(h^2, N_1(h))$ and $(h^2/4, N_1(h/2))$ satisfies $P_{0,1}(0) = N_2(h)$. Similarly, show that $P_{1,2}(0) = N_2(h/2)$.
    b. Show that the linear interpolating polynomial $P_{0,2}(h)$ through $(h^4, N_2(h))$ and $(h^4/16, N_2(h/2$ satisfies $P_{0,2}(0) = N_3(h)$.